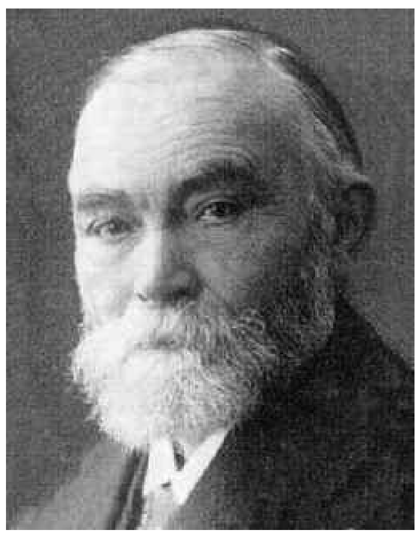
2.2. Russell's Paradox

Note. In this section we give some history of logic and set theory as it relates to the foundations of mathematics. We describe Russell's Paradox and use it as a cautionary tale that motivates a careful approach to the axioms which give the existence of sets.

Note. Around 1900, there was a movement to explore the foundations of mathematics. An attempt to put the foundations on something more fundamental: logic and set theory. Gottlob Frege (November 8, 1848–July 26, 1925) in *Foundations of Arithmetic* (1884) addressed such fundamental questions as "What are number?" and "What is the nature of arithmetical truth?" This work was non-technical and written without symbolism and only gave sketches of proofs. The reaction to this book was not enthusiastic, and even hostile be some. In *The Basic Laws of Arithmetic, Volume 1* (1893) Frege axiomatized arithmetic and gave formal proofs using the axioms of number theory results that, to that time, had only been argued informally (well, "informally" by the standards of the growing movement for rigor at the turn of the 20th century). He planned on dealing with the real numbers in a later volume. While the second volume was at the printers, Frege recieved a letter (on June 6, 1902) from Bertrand Russell in which Russell pointed out that Frege's axioms lead to a contradiction. This contradiction is known as *Russell's Paradox* and is the topic of this brief section. In response, Frege modified one of his axioms in an attempt to give a consistent axiomatic system. However, this change meant that some of the proofs in Volume 1 no longer held. Unfortunately, if was shown later (after Frege's death) that the new system of axioms is also inconsistent. Frege's work influenced that of Giuseppe Peano (who gave a well-known axiomatic system for elementary arithmetic in 1899), Ludwig Wittgenstein (a philosopher with interests in logic the the philosophy of mathematics), and Bertrand Russell. However, this influence was not widespread at the time of its publication, but Frege's influence grew in the second half of the twentieth century after his work was translated into English. These historical notes (and the photo below) are based on the MacTutor History of Mathematics Archive biography of Frege (accessed 12/26/2021).



Gottlob Frege (November 8, 1848–July 26, 1925)

Note. While Frege concentrated more on symbolic logic as a foundation of mathematics, Bertrand Russell (May 18, 1872–February 2, 1970) focused on set theory as a foundation. He introduced his paradox in his 1903 book *Principles of Mathematics* (a copy of which is online at Fair-Use.org). An explanation of how Russell's Paradox came about is explained on the MacTutor History of Mathematics Archive biography of Russell (on which these historical notes about Russell are based and the course of the photo below; accessed 12/26/2021) as follows:

> "Russell's paradox arises as a result of naive set theory's so-called unrestricted comprehension (or abstraction) axiom. Originally introduced by Georg Cantor, the axiom states that any predicate expression, $P(x)$, which contains x as a free variable, will determine a set whose members are exactly those objects which satisfy $P(x)$. The axiom gives form to the intuition that any coherent condition may be used to determine a set (or class). Most attempts at resolving Russell's paradox have therefore concentrated on various ways of restricting or abandoning this axiom."

Russell himself deals with the paradox by introducing his "theory of types," first in an article "Mathematical Logic as Based on the Theory of Types" in 1908 and then in his three volume *Principia Mathematica* (1910, 1912, and 1913) which he coauthored with Alfred North Whitehead.

Bertrand Russell (May 18, 1872–February 2, 1970)

The three volumes of *Principia Mathematica* were published by Cambridge University Press. A copy of the first volume is online at The University of Michigan

Historical Mathematics Collection. The level of detail is so great, that it takes 379 pages before the proof that $1+1=2$ is given:

$*54 \cdot 43$. $\vdash :. \alpha, \beta \in 1 . \supset : \alpha \cap \beta = \Lambda . \equiv . \alpha \cup \beta \in 2$

Dem.

$\vdash . *54 \cdot 26 . \supset \vdash :. \alpha = \iota \text{`} x . \beta = \iota \text{`} y . \supset : \alpha \cup \beta \in 2 . \equiv . x \neq y .$

$[*51 \cdot 231] \qquad \equiv . \iota \text{`} x \cap \iota \text{`} y = \Lambda .$

$[*13 \cdot 12] \qquad \equiv . \alpha \cap \beta = \Lambda \qquad (1)$

$\vdash . (1) . *11 \cdot 11 \cdot 35 . \supset$

$\vdash :. (\exists x, y) . \alpha = \iota \text{`} x . \beta = \iota \text{`} y . \supset : \alpha \cup \beta \in 2 . \equiv . \alpha \cap \beta = \Lambda \qquad (2)$

$\vdash . (2) . *11 \cdot 54 . *52 \cdot 1 . \supset \vdash .$ Prop

From this proposition it will follow, when arithmetical addition has been defined, that $1+1=2$.

From page 379 of *Principia Mathematica*

Note. Before formally stating Russell's Paradox, we tell a story that illustrates it. Imagine a town with a barber. The barber cuts the hair of all of those who do not cut their own hair. We ask: "Who cuts the barber's hair?" If the barber does not cut their own hair, then the barber must cut their own hair (since that is their job). If the barbers does cut their own hair, then they cannot cut their own hair since their job is to cut the hair of those who do not cut their own hair. Another, more set theoretic, description is to consider the set S of all sets that are not members of themselves. The question then is: "Is set S a member of itself?" If S is a member of itself, then it cannot me a member of itself since it only consists of such sets. If S is not a member of itself, then it must be a member of itself by its own definition. Therefore such a set cannot exist.

Note. We now explore Russell's Paradox as presented by Gerstein. We start with a property P and *assume* that the property can be used to define a set: $\{x \mid P(x)\}$. Consider the set

$$S = \{A \mid A \text{ is a set and } A \notin A\}.$$

Notice that some sets are not elements of themselves (for example, the set of integers $\mathbb{Z}$ does not include the set $\mathbb{Z}$ itself) and some sets are elements of themselves (for example consider the set of sets $A = \{A, B, C\}$). We ask the question: "Is the set S under discussion a member of itself?" If $S \in S$ (so that the proposition P is violated) then we cannot have S as an element of S. Therefore, we must have $S \notin S$. But if $S \notin S$ then property P is satisfied and hence we must have $S \in S$, also an impossibility!

Note. The analogy with the barber story is that we take the individuals of the town as sets. We indicate the relationship "$A \notin A$" as "A does not cut their own hair." Being in set S means that one does not cut one's own hair (so we would think of proposition P as being an individual in the town who does not cut their own hair); that is, S is the set of individuals in the town for whom the barber cuts their hair. We ask if the barber cuts their own hair or not, we are asking if the barber is in set S or not (that is, the $A \in A$ or $A \notin A$ where A represents the barber).

Note. The resolution of the paradox is that we abandon the notion that S is a set! So some modification of the use of a property P to define a set as $\{x \mid P(x)\}$. Notice that the Axiom of Separation of Section 2.1. Fundamentals starts with the existence of a set X and uses the proposition P to create a subset of X: $\{x \mid x \in X \text{ and } P(x)\}$. In Russell's Paradox, we are considering the somewhat ambiguous collection of "all sets" (when we say "A is a set"). An axiomatic approach to set theory (as opposed to a naive approach) must be more precise in what is allowed to be a "set." For example, there can be no "set of all sets," otherwise we *would* have by the Axiom of Separation that $S = \{A \mid A \in \mathscr{X} \text{ and } A \not\in A\}$ is a set (where $\mathscr{X}$ is the "set of all sets"). But the existence of S as a set then leads to the contradiction given by Russell's Paradox.

Example 2.7. We now explore the correct use of the Axiom of Separation in a setting similar to Russell's Paradox. Suppose that X is a set. Then

$$S = \{A \in X \mid A \text{ is a set and } A \not\in A\}$$

is a set by the Axiom of Separation. Again we ask: "Is S a member of itself?" If $S \in S$ then the description of set S implies that $S \in X$, S is a set, and $S \not\in S$. But this is a contradiction since we require both $S \in S$ and $S \not\in S$. So the hypothesis $S \in S$ is false and we must have $S \not\in S$. Now the description of set S again implies that its elements are elements of set X, are themselves sets, and are not elements of themselves. Since we have concluded that set S satisfies $S \not\in S$, then the one condition on the elements of S that S must violate is that $S \not\in X$. This conclusion is not a paradox, since we nowhere assumed that $S \in X$. (Subtle stuff, eh?)

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