2.3. Quantifiers

Note. In this section we introduce a symbolic notation that will be used in the rest of this chapter (and probably throughout your mathematical education and career). No new theorems are produced, but instead we focus on using the symbols and translating between verbal statements and statements made using the new symbols.

Definition. A *variable* x is a symbol used in a statement P such that P becomes a proposition when the symbol is replaced with a specific element of some set.

Note. For example, in statement $P(x)$ of "$x^2 + 8 = 17$" is not a proposition since variable x is not given a value. One a value is assigned to x the $P(x)$ is a proposition with a truth value depending on the value of x.

Note/Definition. We will often want to claim that there is at least one element for variable x which makes a proposition true. We denote this as "$(\exists x)P$" and read it as "There exists an x such that P" or "For some x, P." The symbol $\exists$ is the *existential quantifier*. We will also want to claim that every element x (of some given set, or given a "universal set") has property P. We denote this as "$(\forall x)P$" and read it as "For all x, P" or "For every x, P." The symbol $\forall$ is the *universal quantifier*.

Definition. In addressing a proposition $P(x)$, if a set is specified from which all elements are to come the this set is the *universe* or *universal set* (or *domain of interpretation*) for the proposition.

Example 2.8. (a) The statement

$$(\exists x)(x \in \mathbb{R} \text{ and } x^2 + \pi x - 2\pi^2 = 0)$$

is a true proposition since a value of x (which is a real number) that makes $P(x) : x^2 + \pi x - 2\pi^2 = 0$ true (namely, $x = \pi$). We may abbreviate the statement as

$$(\exists x \in \mathbb{R})(x^2 + \pi x - 2\pi^2 = 0)$$

or, if we know the universal set is $\mathbb{R}$, as $(\exists x)(x^2 + \pi x - 2\pi^2 = 0)$.

(c) A well-known theorem states that every nonnegative real number has a square root (it actually follows from the definition of the real numbers; see my online notes for Analysis 1 [MATH 4217/5217] on Section 1.3. The Completeness Axiom and notice the definition of x^r for $x > 0$ and r rational). This can be written as the two-variable proposition

$$(\forall x)(x \in \mathbb{R} \wedge x \geq 0 \Rightarrow (\exists y)(y \in \mathbb{R} \wedge y^2 = x)).$$

If we understand the universal set to be $\mathbb{R}$ then we can abbreviate the proposition to

$$(\forall x)(x \geq 0 \Rightarrow (\exists y)(y^2 = x)).$$

Notice that we interpret this as x, being a nonnegative real number, has some real number square root y (and notice that this is stated in terms of the square of y, not in terms of a square root of x). If we mix words and symbols (for clarity, really) then we can write

$$\forall x \geq 0, \exists y \text{ such that } y^2 = x$$

or more compactly and symbolically (but maybe not as clearly) as:

$$(\forall x \geq 0)(\exists y)(y^2 = x).$$

Note. We now consider the interaction of negation $\sim$ of the types of statements given above. Notice that $(\forall x)P$ is equivalent to $\sim (\exists x)(\sim P)$ (which we might read as "there *does not exist* x such that not P"); in fact, this is how Gerstein introduces the universal quantifier. This gives the equivalences

$$\sim (\forall x)P \equiv \sim\sim (\exists x)(\sim P) \equiv (\exists x)(\sim P)$$

(which, when converted to words, clearly holds). Similarly,

$$\sim (\exists x)P \equiv (\exists x)(\sim (\sim P)) - \equiv (\forall x)(\sim P).$$

So we have equivalences related to negating the quantifiers as:

$$\sim (\exists x)P \equiv (\forall x)(\sim P) \tag{2.9}$$

$$\sim (\forall x)P \equiv (\exists x)(\sim P) \tag{2.10}$$

Example 2.11. We now give some examples concerning translations of symbolic statements into sentences, and vice versa. We use the first letter of a word as the variable representing it.

(a) "There is a cat in that house" can be represented as $(\exists c)(c \text{ is in that house})$. By (2.9), the negation is $(\forall c)(c \text{ is outside that house})$ which translates into "Every can is outside that house."

(b) "Every animal eats some food" can be translated as $(\forall a)(\exists f)(a \text{ eats } f)$. By (2.10) the negation can be written $(\exists a)(\sim (\exists f)(a \text{ eats } f))$ and by (2.9) can be written $(\exists a)(\forall f)(a \text{ does not eat } f)$ which translates into "There is an animal that eats no food."

(c) "There is an animal that eats every food" can be translated as $(\exists a)(\forall f)(a \text{ eats } f)$. Similar to part (b), this negates to $(\forall a)(\exists f)(a \text{ does not eat } f)$ which translates as "For every animal there is some food it does not eat."

(e) "You can fool all of the people all of the time" can be translated as

$$(\forall p)(\forall t)(\text{You can fool } p \text{ at time } t).$$

By (2.10) the negation can be written

$$\sim ((\forall p)(\forall t)(\text{You can fool } p \text{ at time } t)) \equiv (\exists p)(\sim (\forall t)(\text{You can fool } p \text{ at time } t))$$

$$\equiv (\exists p)(\exists t)(\sim (\text{You can fool } p \text{ at time } t)) \equiv (\exists p)(\exists t)(\text{You can't fool } p \text{ at time } t),$$

which translates as "There exists a person and a time when the person can't be fooled" or, as Gerstein puts it, "There is a person who sometimes can't be fooled."

Note/Definition. We will often want to claim that there exists a unique element for variable x which makes a proposition true. We denote this as "$(\exists! x)P$" and read it as "There exists a unique x such that P" or "There exists exactly one x, P." The symbol $\exists!$ is the *uniqueness existential quantifier*.

Note. To establish a proposition of the form $(\exists! x)P$ we must show (1) existence, $(\exists x)P$, and (2) uniqueness in which we prove that if P is true for both x and y then necessarily $x = y$.

Exercise 2.3.9(c). Suppose the underlying universal set is $\mathbb{R}$. Consider the statement

$$(\forall x)(\exists! y)(y^3 = x).$$

Is this true or false?

Solution. The statement translates to "Every real number x has a unique real cube root." This is true. Notice that we would have statement $(\forall x)(\exists! y)(y^2 = x)$ false since negative real numbers do not have (real) square roots and positive real numbers have two possible values for y, namely $y = \pm\sqrt{x}$. If the underlying universal set were $\mathbb{C}$ in $(\forall x)(\exists! y)(y^3 = x)$ then the result would be false since every nonzero complex number has three complex cube roots. □

Revised: 12/26/2021