2.4. Set Inclusion

Note. In this section we define the relationship of "subset" between two sets. With this new idea, and the background we have established, we are ready to prove several results.

Definition 2.12. Let A and B be sets. Set A is a *subset* of set B (or A is *contained* in B, or B *contains* A, of B *includes* A), written $A \subseteq B$ (or $B \supseteq A$) if every member of A is a member of B. That is,

$$A \subseteq B \Leftrightarrow (\forall x)(x \in A \Rightarrow x \in B).$$

The symbol "$\subseteq$" is the *set inclusion* symbol. If A is not a subset of B, then we write $A \not\subseteq B$.

Note. For example, we have the subset inclusions $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$ (this is Example 2.13(c) in the book).

Note. The use of the set inclusion symbol is not universal. Sometimes it is replaced with the symbol "$\subset$." Sometimes this symbol is used to represent a "proper subset"; that is, $A \subset B$ sometimes means A is a subset of B and $A \neq B$. A clearer notation is to use the set inclusion symbol "$\subseteq$" as Gerstein has introduced it, but to use the symbol "$\subsetneq$" to indicate a proper subset. This is addressed below in Definition 2.19.

Note. To prove that $A \subseteq B$ it is sufficient to show that $x \in A \Rightarrow x \in B$ (this is a direct proof). We can also consider the contrapositive of this implication to give an indirect proof: $x \notin B \Rightarrow x \notin A$. To prove that $A \not\subseteq B$, it is sufficient to show that A contains an element that is not in B. This claim can be justified with the following argument:

$$\begin{aligned} A \not\subseteq B &\Leftrightarrow \sim (\forall x)(x \in A \Rightarrow x \in B) && \text{by Definition 2.12} \\ &\Leftrightarrow (\exists x)(\sim (x \in A \Rightarrow x \in B)) && \text{by (2.10)} \\ &\Leftrightarrow (\exists x)(x \in A \text{ and } x \notin B) && \text{from a truth table.} \end{aligned}$$

Theorem 2.14. Let A be a set. Then $A \subseteq A$ and $\varnothing \subseteq A$.

Note. Gerstein gives a detailed argument "with reasoning exposed" of the following. We present a largely symbolic argument, with some of the reasoning exposed.

Theorem 2.15. If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.

Theorem 2.17. $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$.

Definition 2.19. For sets A and B, define the *proper inclusion* relation (or *strict inclusion*), denoted either $\subset$ or $\subset\neq$, as:

$$A \subset B \Leftrightarrow A \subseteq \text{ and } A \neq B.$$

We then have that A is a *proper subset* of B (or A is *properly contained* in B).

Note. To prove that A is a proper subset of B, it is sufficient to prove that $A \subseteq B$ and that there is some $x \in B$ such that $x \notin A$. If A is not a proper subset then Gerstein writes $A \not\subset B$. This is a bit unconventional since this notation allows for the symbols $A \not\subset B$ to include the case $B = A$. We'll try to follow Gerstein's notation, but will make sure we are unambiguous when a distinction must be drawn between a subset and a proper subset.

Exercise 2.4.8. Let A and B be sets. Prove that $A \subseteq B$ if and only if every subset of A is a subset of B.

Revised: 12/27/2021