2.5. Union, Intersection, and Complement

Note. In this section we define, as the title of the section suggests, unions, intersections, and complements of sets. We illustrate these ideas with Venn diagrams and state several standard results (and prove some, with proofs of others left as exercises).

Definition 2.21. Let A and B be sets. The *complement of B in A* is

$$A - B = \{x \in A \mid x \notin B\},$$

also called the *difference* of the sets of the *relative complement of B in A*. This is also often denoted $A \setminus B$. If U is a given universal set, the complement of A in U is the *complement* of A, denoted A'. (Sometimes A complement is denoted A^c, but this is not universal and these symbols are sometimes used for the "closure" of A).

Example 2.22, Exercise 2.23. Some elementary properties of differences and complements of sets are the following.

2.22(a) For $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$, $A - B = \{1\}$ and $B - A = \{4\}$.

2.22(b) $\{\varnothing, \{\varnothing\}\} - \{\varnothing\} = \{\{\varnothing\}\}$.

2.23(a) $A - A = \varnothing = \varnothing - A$.

2.23(b) $A - \varnothing = A$.

2.23(c) $A - B = B - A \Leftrightarrow A = B$.

2.23(d) If U is the universal set, then $(A')' = A$, $\varnothing' = U$, and $U' = \varnothing$.

Definition 2.24. The *union* of sets A and B is the set

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

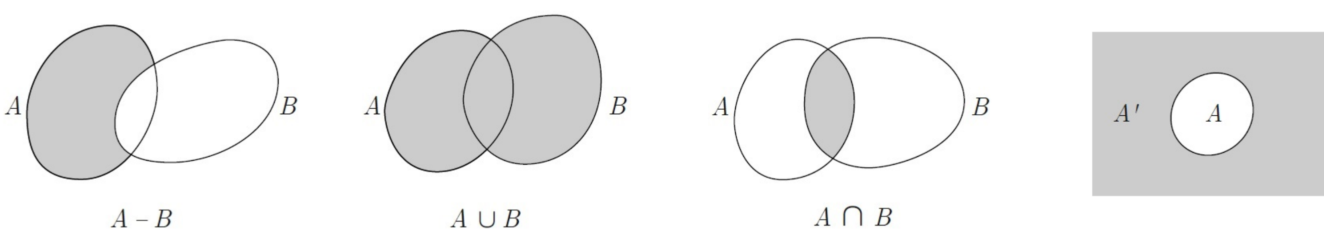
The *intersection* of A and B is the set

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

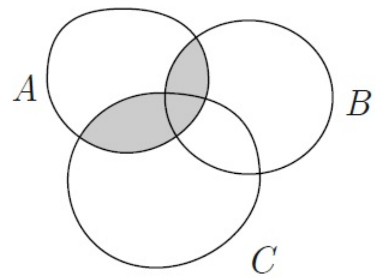
Example 2.25. Let $A = \{1, 2, 3, 4\}$, $B = \{0, 1, 3, 5, 7\}$, and $C = \{2, 4, 6, 8\}$. Then

$A \cup B = \{0, 1, 2, 3, 4, 5, 7\}$, $A \cap B = \{1, 3\}$, $B \cap C = \varnothing$, $(A \cup B) \cap C = \{2, 4\} = A \cap C$.

Note. A common way to illustrate the above ideas is to use regions in the plane to illustrate sets and to shade the regions which represent unions, intersections, or complements. Such drawings are called "Venn diagrams." These were popularized by John Venn (August 4, 1834–April 4, 1923), an English mathematician and logician, in Chapter V "Diagrammatic Representation" of his 1881 book *Symbolic Logic* (a copy of which in online at Archive.org; see pages 100–125). Gerstein presents the following Venn diagrams (form pages 59 and 60):



As an example of the use of Venn diagrams, we can illustrate the set $(A \cap B) \cup (A \cap C)$ as follows (from Gerstein, page 60):



Venn diagrams may be useful in getting ideas about how to develop a proof (or to illustrate a proof that you are trying to read), but they cannot be used in place of a logical proof!

Theorem 2.16. Let A, B, C be sets and let U be the universal set. The following hold.

$$\left.\begin{array}{ll}\text{(a)} & A \cap (B \cap C) = (A \cap B) \cap C \\ \text{(b)} & A \cup (B \cup C) = (A \cup B) \cup C\end{array}\right\} \quad \text{associative laws}$$

$$\left.\begin{array}{ll}\text{(c)} & A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \\ \text{(d)} & A \cup (B \cap C) = (A \cup B) \cap (A \cup C)\end{array}\right\} \quad \text{distributive laws}$$

$$\left.\begin{array}{ll}\text{(e)} & A - (B \cup C) = (A - B) \cap (A - C) \\ \text{(f)} & A - (B \cap C) = (A - B) \cup (A - C) \\ \text{(g)} & (A \cup B)' = A' \cap B' \\ \text{(h)} & (A \cap B)' = A' \cup B'\end{array}\right\} \quad \text{De Morgan's laws}$$

Theorem 2.27. (a) If $X \subseteq Z$ and $Y \subseteq Z$ then $X \cup Y \subseteq Z$.

(b) If $Z \subseteq X$ and $Z \subseteq Y$ then $Z \subseteq X \cap Y$.