2.6. Indexed Sets

Note. In this section we consider collections of sets indexed by elements from an indexing set. We give examples, define unions and intersections of indexed sets, and state some set equalities related to these new ideas.

Note/Definition. We can describe a set by associating its elements with members of an *index set*. We can write $B = \{b_i \mid i \in I\}$ where set I is the index set. For example, if $I = \{1, 2, 3, 4\}$ then set B is $B = \{b_1, b_2, b_3, b_4\}$. The index set need not be finite, though. We could consider the set

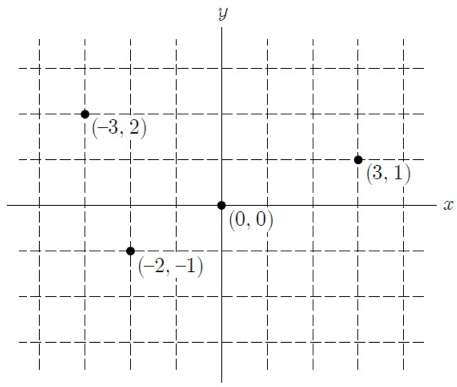
$$S = \{s_i = i^2 \mid i \in \mathbb{N}\} = \{1, 4, 9, 16, \ldots\}.$$

In fact, the index set need not even be countable:

$$R = \{r_x = x^2 \mid x \in \mathbb{R}\}.$$

We will explore cardinalities and different "kinds" of infinities in Chapter 4, "Finite and Infinite Sets."

Example 2.28(c). We denote the plane $\mathbb{R}^2$ as Π. Treating Π as a set of points, we can index the points of Π with ordered pairs of real numbers (as René Descartes did in his introduction of analytic geometry in 1637 in *La G'eométrie*). This requires the introduction of two perpendicular axes, then the measuring of distances of points from the axes (with the usual choice of sign based on quadrants). The next illustration is from page 64 of the text book:



We can then index any point $P \in \Pi$ with an ordered pair (x, y) of real numbers, so that the index set is $I = \{(x, y) \mid x, y \in \mathbb{R}\} = \{(x, y) \mid (x, y) \in \mathbb{R}^2\}$ and the Cartesian plane is $\Pi = \{P_{(x,y)} \mid (x, y) \in \mathbb{R}^2\}$. Alternatively, we could use the polar coordinates of a point in Π as the index. The index set then becomes $I = \{(r, \theta) \mid r \in \mathbb{R}, r \geq 0, \theta \in \mathbb{R}, 0 \leq \theta < 2\pi\}$ and the Cartesian plane is $\Pi = \{P_{(r,\theta)} \mid r \in \mathbb{R}, r \geq 0, \theta \in \mathbb{R}, 0 \leq \theta < 2\pi\}$.

Example 2.29(b). We can also consider a set of subsets of Π and index these with certain real numbers. For every $r \in \mathbb{R}$ we define a right half-plane as $H_r = \{P_{(x,y)} \mid (x, y) \in \mathbb{R}^2, x \geq r\}$. The set of all right half-planes is then $H = \{H_r \mid r \in \mathbb{R}\} = \{H_r\}_{r \in \mathbb{R}}$. If r is real and $r \geq 0$ we define a circle with radius r and center $(0, 0)$ as $C_r = \{P_{(x,y)} \mid \sqrt{x^2 + y^2} = r\}$ (notice that this definition admits the single point $(0, 0)$ as a circle of radius 0; this is usually called a "degenerate" circle). The set of all such circles is then $C = \{C_r \mid r \in \mathbb{R}, r \geq 0\} = \{C_r\}_{r \geq 0}$. Similarly, we can define a closed disk with radius r and center $(0, 0)$ as $D_r = \{P_{(x,y)} \mid \sqrt{x^2 + y^2} \leq r\}$ and the set of all such disks is then $D = \{D_r \mid r \in \mathbb{R}, r \geq 0\} = \{D_r\}_{r \geq 0}$.

Note. We introduced unions and intersections of *two* sets at a time in the last section. We now extend these ideas to indexed sets over an arbitrary indexing set.

Definition 2.30. Let $\{A_i\}_{i\in I}$ be an indexed family of sets. Define the *union* $\cup_{i\in I}A_i$ as the set of elements that belong to at least one A_i. Define the *intersection* $\cap_{i\in I}A_i$ as the set of elements that belong to all A_i.

Note. We can more symbolically present unions and intersections over an index set as:

$$\cup_{i\in I}A_i = \{x \mid x \in A_i \text{ for some } i \in I\} = \{x \mid (\exists i \in I)(x \in A_i)\}$$

and

$$\cap_{i\in I}A_i = \{x \mid x \in A_i \text{ for all } i \in I\} = \{x \mid (\forall i \in I)(x \in A_i)\}.$$

Note 2.6.A. To say that $x \notin \cup_{i\in I}A_i$ is to say $\sim (\exists i \in I)(x \in A_i)$ or (by (2.9) in Section 2.3) $(\forall i \in I)(x \notin A_i)$. To say that $x \notin \cap_{i\in I}A_i$ is to say $\sim (\forall i \in I)(x \in A_i)$ or (by (2.10) in Section 2.3) $(\exists i \in I)(x \notin A_i)$.

Example 2.31(c) Consider again the half-planes, circles, and disks of Example 2.29(b). In this example, we do not distinguish between a point P in the plane Π and its index (x, y) (so the elements of π and of half-plane/circles/closed disks are all ordered pairs of real numbers; with this notation, Gerstein denotes the plane as E so we also follow this notation). We first claim that $\cup_{r\in\mathbb{R}}H_r = E$. By definition, we have $H_r \subseteq E$ for every $r \in \mathbb{R}$. For any $(a,b) \in \cup_{r\in\mathbb{R}} H_r$ we have $(a,b) \in H_{r'}$ for some $r' \in \mathbb{R}$ by the definition of union. Since $H_{r'} \subseteq E$ then $(a,b) \in E$. Since (a,b) is an arbitrary element of $\cup_{r\in\mathbb{R}} H_r$ then we have $\cup_{r\in\mathbb{R}} H_r \subseteq E$. Next, for any $(a,b) \in E$ we have $(a,b) \in H_{a-1}$ by the definition half-plane. So $(a,b) \in \cup_{r\in\mathbb{R}} H_r$ and and since (a,b) is an arbitrary element of E then $E \subseteq \cup_{r\in\mathbb{R}} H_r$. So by Theorem 2.17 we have $\cup_{r\in\mathbb{R}} H_r = E$, as claimed. Notice that $\cap_{r\in\mathbb{R}} H_r = \varnothing$ since for any $(a,b) \in E$ we have $(a,b) \notin H_{a+1}$ and hence $(a,b) \notin \cap_{r\in\mathbb{R}} H_r$. We next claim that $\cup_{r\geq 0} C_r = E$.

Note. We conclude this section with several claims that generalize Theorem 2.26 to indexed unions, intersections, and complements.

Theorem 2.32. Let A be a set, $\{B_i\}_{i\in I}$ be an indexed family of sets, and let U be the universal set. Then:

(a) $A - \cap_{i\in I} B_i = \cup_{i\in I}(A - B_i)$

(b) $A - \cup_{i\in I} B_i = \cap_{i\in I}(A - B_i)$

(c) $(\cap_{i\in I} B_i)' = \cup_{i\in I} B_i'$

(d) $(\cup_{i\in I} B_i)' = \cap_{i\in I} B_i'$

Note. The proofs of parts (c) and (d) in Theorem 2.32 follows easily from parts (a) and (b) by taking set A to be the universal set U (since $U - B = B'$ for every set B).