2.7. The Power Set

Note. In this section we consider the set of all subsets of a given set. Following our concerns over Russell's Paradox in Section 2.2, we should be worried about just stating that such a set exists. In an axiomatic development of set theory, we would take this as an axiom; see my online notes for such a class on Section 1.3. The Axioms (notice The Axiom of Power Set). We introduce decision trees in this section when we informally discuss the subset of a given finite set.

Definition 2.33. The *power set* $P(A)$ of a set A is the collection of all subsets of A: $P(A) = \{X \mid X \subseteq A\}$.

Example 2.34. Notice that $\varnothing$ is a subset of every set (by Theorem 2.14), so $\varnothing \in P(A)$ for every set A.

(a) $P(\{3\}) = \{\varnothing, \{3\}\}$

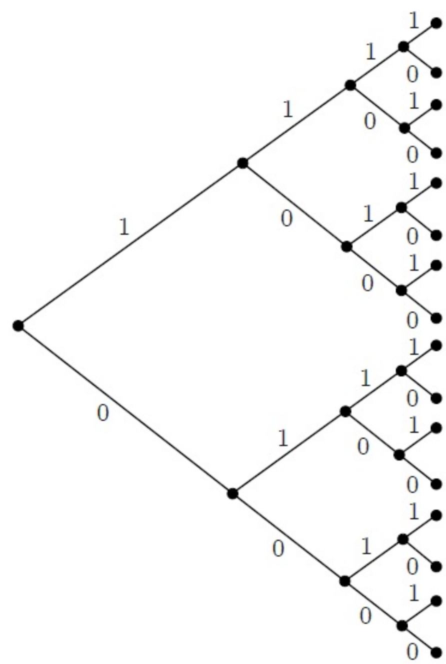
(b) $P(\{1,2\}) = \{\varnothing, \{1\}, \{2\}, \{1,2\}\}$

(c) $P(\varnothing) = \{\varnothing\}$.

(d) (Modified) $P(\{1,2,3\}) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3,\}\}$

Note. We will see later that if A has n elements then $P(A)$ has 2^n elements. This will be proved by induction in Section 2.10. Mathematical Induction and Recursion (see Theorem 2.69). Notice that this is consistent with Example 2.34.

Note/Definition. When looking for the power set of a given finite set, we can produce a subset by making a decision of whether to (1) include or (2) exclude every particular element of the set. Intuitively, this is *why* a set with n elements has a power set with 2^n elements. We can use a *decision tree* to illustrate this (Figure (2.35) in Gerstein):



The points are called *vertices* (sometimes "nodes"), the lines are called *edges* (sometimes "branches"), and a vertex at the end of exactly on edge is a *leaf*. A tree is a special case of a more general structure called a *graph*. These structures are discussed in more detail in Section 5.9. Graphs, and are the topic of ETSU's "Introduction to Graph Theory" (MATH 4357/5357) (I have online notes for this class). In the decision tree, a column of 1s and 0s corresponds to the presence or absence of a particular element of the set in the subset; a 1 represents inclusion of an element and 0 represents exclusion of the element. To find a particular subset, we start with the left-most vertex (called the *root* of the tree) and read along some path, always moving to the right, until we reach a right-most vertex. This produces a sequence of four 1s and 0s. For example, the sequence 1010 corresponds to a subset that contains the first and third elements and excludes the second and fourth elements. In the figure, there are 16 leaves on the right, corresponding (reading from bottom to top) to the sequences 0000, 0001, 0010, . . . , 1110, 1111 (the binary representations of the numbers 0 through 15). We now state some relationships between the power sets of two given sets.

Theorem 2.36. Let A and B be sets. Then:

(a) $\{\varnothing, A\} \subseteq P(A)$

(b) $A \subseteq B \Leftrightarrow P(A) \subseteq P(B)$

(c) $P(A) \cup P(B) \subseteq P(A \cup B)$

(d) $P(A) \cap P(B) = P(A \cap B)$

Exercise 2.7.8. Let A be a set, and suppose $x \notin A$. Describe $P(A \cup \{x\})$.

Revised: 12/29/2021