2.8. Ordered Pairs and Cartesian Products

Note. We encountered ordered pairs in Section 2.6. Indexed Sets in connection with the Cartesian plane. In this section we formalize the idea of ordered pairs. This idea is generalized later in Section 4.1. Cardinality; Fundamental Counting Principles where we extend the idea of ordered pairs to ordered n-tuples (an idea you like encountered in Linear Algebra [MATH 2010]). See also Exercise 2.8.6, which concerns ordered triples and ordered quadruples.

Note. Gerstein presents a "quasi-definition" or ordered pairs as (see page 76): "Let a and b be elements. The symbol (a, b) denoted an *ordered pair*, with the following understanding: if (c, d) is also an ordered pair, then

$$(a, b) = (c, d) \Leftrightarrow a = c \text{ and } b = d.''$$

But notice that this does not actually *define* an ordered pair, but only introduces a notation and a property of "ordered pairs." Since this chapter deals with a careful development of set theory (remember Russell's paradox), we need to deal with this concept in terms of sets.

Definition 2.39. Let a and b be elements. Define $(a, b) = \{\{a\}, \{a, b\}\}$. The elements a and b are called the *first* and *second coordinates* of (a, b), respectively.

Note. See my online notes for "Set Theory" (which concern a careful, axiomatic development) on Section 2.1. Ordered Pairs. The definition of *ordered pair* is the same as what we have here. Also, the "quasi definition" appears in those notes as a theorem (Theorem 2.1.2 in those notes). We have the same situation here, as the next theorem shows.

Theorem 2.40. $(a, b) = (c, d) \Leftrightarrow a = c$ and $b = d$.

Corollary 2.41. $(a, b) = (b, a) \Leftrightarrow a = b$.

Example 2.42(b). Suppose a collection of numerical data is stored in a computer's memory. If the number m is stored in memory cell n, then we have the set

$$T = \{(m, n) \mid m \text{ is stored at address } n\}.$$

Set T is a complete record of what numbers are where. Of course the order matters in avoiding confusion between a location n and a value m.

Definition 2.43. The *Cartesian product* of sets A and B, denoted $A \times B$, is the set of all ordered pairs with first coordinate in A and second coordinate in B:

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}.$$

Example 2.44. **(a)** Let $A = \{1, 2\}$ and $B = \{2, 3\}$. Then

$$A \times B = \{(1, 2), (1, 3), (2, 2), (2, 3)\} \text{ and } B \times A = \{(2, 1), (2, 2), (3, 1), (3, 2)\}.$$

So $A \times B \neq B \times A$ (not surprisingly).

(b) The Cartesian product $\mathbb{R} \times \mathbb{R}$ is the set of all ordered pairs of real numbers. As in Section 2.6. Indexed Sets, we do not in practice distinguish between a point in the Cartesian plane and the ordered pair of real numbers which *represents* that point (or was the index of the point in Section 2.6). We alos denote $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$.

(d) In $\mathbb{R} \times \mathbb{R}$, consider the circle C_1 with radius 1 centered at the origin $(0, 0)$. Then we have the set

$$C_1 \times \mathbb{R} = \{((x, y), z) \mid z\, y, z \in \mathbb{R} \text{ and } x^2 + y^2 = 1\}.$$

Notice that the elements of C_1 are themselves ordered pairs. We can associate an element $((x, y), z) \in C_1 \times \mathbb{R}$ with the point (x, y, z) in three-dimensional space. We then have the geometrical representation of the set as an infinitely long cylinder of radius 1 with axis as the z-axis.

Theorem 2.45. Let A, B, and C be sets. Then:

(a) $(A \cup B) \times C = (A \times C) \cup (B \times C)$

(b) $(A \cap B) \times C = (A \times C) \cap (B \times C)$

(c) $(A - B) \times C = (A \times C) - (B \times C)$

(d) If A and B are nonempty sets then $A \times B = B \times A \Leftrightarrow A = B$.

(e) If $A_1 \in P(A)$ and $B_1 \in P(B)$, then $A_1 \times B_1 \in P(A \times B)$.

(f) If A and B each have at least two elements, then not every element of $P(A \times B)$ has the form $A_1 \times B_1$ for some $A_1 \in P(A)$ and $B_1 \in P(B)$.

(g) $\varnothing \times A = \varnothing$

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