2.9. Set Decomposition: Partitions and Relations

Note. In this section we introduce an idea fundamental to mathematics, that of an equivalence relation. Equivalence relations play a big role in number theory and modern algebra, in particular. We will show that an equivalence relation on a set determines a partition and, conversely, a partition of a set determines an equivalence relation.

Definition 2.47. Let S be a nonempty set. A *partition* Π of S is a family $\Pi = \{A_i\}_{i\in I}$ of nonempty subsets of S satisfying these conditions:

(1) $\cup_{i\in I} A_i = S$.

(2) $A_i \cap A_j = \varnothing$ if $i \neq j$.

The A_i are *blocks* of the partition.

Example 2.48. (a) Consider set $S = \{1, 2, 3, 4\}$. There is one partition of S with 1 block: $\{\{1, 2, 3, 4\}\}$. There are seven partitions of S with 2 blocks:

$$\begin{array}{ccc} \{\{1\},\{2,3,4\}\} & \{\{2\},\{1,3,4\}\} & \{\{3\},\{1,2,4\}\} \\ \{\{4\},\{1,2,3\}\} & \{\{1,2\},\{3,4\}\} & \\ \{\{1,3\},\{2,4\}\} & \{\{1,4\},\{2,3\}\} & \end{array}$$

There are six partitions of S with 3 blocks:

$$\begin{array}{ccc} \{\{1\},\{2\},\{3,4\}\} & \{\{1\},\{3\},\{2,4\}\} & \{\{1\},\{4\},\{2,3\}\} \\ \{\{2\},\{3\},\{1,4\}\} & \{\{2\},\{4\},\{1,3\}\} & \{\{3\},\{4\},\{1,2\}\} \end{array}$$

There is one partition of S with 4 blocks: $\{\{1\}, \{2\}, \{3\}, \{4\}\}$.

(d) In the Cartesian plane E (where we have associated points and the ordered pair of real numbers representing them), the family $\{L_i\}_{i\in\mathbb{R}}$ (where L_i represents the vertical line in E of the form $y = i$) is a partition of E. The family $\{C_r\}_{r\geq 0}$ of circles with center $(0,0)$ and radius r (see Example 2.29(b)) is also a partition of E. We also considered closed disks D_r of radius $r \geq 0$ and center $(0,0)$, $\{D_r\}_{r\geq 0}$, but the intersection of any two such disks is nonempty and so $\{D_r\}_{r\geq 0}$ does not partition E.

Example 2.49. (a) We will often construct partitions of a set which involve blocks that contain elements with a "common feature." An example foreshadowing this approach involves the integers. The oddness or evenness of an integer is its *parity*. The relation of "having the same parity" leads to a partition of $\mathbb{Z}$ into two blocks, the set of even integers and the set of odd integers. We now formally define a relation.

Definition 2.50. A *relation* R on a set S is a collection of ordered pairs of elements of S; that is, a subset $R \subseteq S \times S$. The assertion $(x,y) \in R$ is denoted xRy and we say "x is *related to* y by R. If $(x,y) \notin R$ we write $x\not R y$. If $R = S \times S$ then R is the *universal relation* on S. If $R = \varnothing$ then R is the *empty relation*.

Example 2.51. **(d)** Let L be the set of all living people. Define relations P and C as:

$$\begin{aligned} P &= \{(x, y \mid x \text{ is a parent of } y\} \\ C &= \{(y, y \mid y \text{ is a child of } x\}. \end{aligned}$$

We then have $xPy \Leftrightarrow yCx$. We say that P and C are *inverse relations*, denoted $P = C^{-1}$ and $C = P^{-1}$.

Definition 2.52. Let R be a relation on the set S. Then R is *reflexive* if xRx for all $x \in S$. The relation R is *symmetric* if for all $x, y \in S$ we have $xRy \Rightarrow yRx$. Relation R is *transitive* if for all $x, y, z \in S$, we have $(xRy$ and $yRz) \Rightarrow xRz$.

Example 2.53. **(b)** Let $S = \{\text{rock}, \text{scissors}, \text{paper}\}$. Define a relation B (read "beats") on S by

$$B = \{(\text{rock}, \text{scissors}), (\text{scissors}, \text{paper}), (\text{paper}, \text{rock})\}.$$

Notice that relation B is neither reflexive, symmetric, nor transitive.

(c) Let $S = \{1, 2, 3\}$ and consider the relations on S of:

$$\begin{aligned} R_1 &= \{(1,1), (2,2), (3,3), (1,2), (2,3)\} \\ R_2 &= \{(1,2), (2,3), (1,3)\} \\ R_3 &= \{(1,2), (2,1)\} \end{aligned}$$

The R_1 is reflexive, but not symmetric (since $1R_12$ and $2\not R_11$), and not transitive (since $1R_12$ and $2R_13$ but $1x\not R_13$). Relation R_2 is transitive by neither reflexive

($1 \not R_2 1$) nor symmetric $1R_2 2$ but $2 \not R_2 1$). To establish transitivity of R_2, we need to verify that $x, y, z \in S$ that the implication (xR_2y and yR_2z) $\Rightarrow xR_2z$ holds. The only values for which (xR_2y and yR_2z) hold are $x = 1$, $y = 2$, and $z = 3$, and we do have xR_2z or $1R_2 3$. Relation R_3 is symmetric but not reflexive since $1 \not R_3 1$ and not transitive because $aR_3 2$ and $2R_3 1$ but $1 \not R_3 1$.

Example. In Example 2.51(c), the relation "$<$" denotes the usual less-than relation on the set $\mathbb{R}$ of real numbers. Formally, we have $(a, b) \in <$ for $a, b \in \mathbb{R}$ if $a < b$. Notice that $<$ is neither reflexive nor symmetric, but it is transitive. If we take the relation "$\leq$" ("less-than-or-equal-to") on $\mathbb{R}$ then we have a reflexive, symmetric, and transitive relation on $\mathbb{R}$.

Definition 2.55. A relation is an *equivalence relation* if it is reflexive, symmetric, and transitive. If $\sim$ is an equivalence relation and $x \sim y$ then we say x and y are *equivalent* with respect to $\sim$.

Example 2.56. **(b)** Consider set $S = \{1, 2, 3, 4\}$ with relation

$$R = \{(1, 2), (2, 1), (1, 1), (2, 2), (3, 3), (4, 4)\}.$$

Then R is reflexive, symmetric, and transitive; that is, R is an equivalence relation on S. Also, $\leq$ is an equivalence relation on $\mathbb{R}$ (probably the equivalence relation that you are most familiar with; of course, $\geq$ is also an equivalence relation on $\mathbb{R}$).

Definition 2.57. If $\sim$ is an equivalence relation on a set S, the set of all elements of S that are related (with respect to $\sim$) to a given element x constitute the *equivalence class* of x, denoted $[x]$. Symbolically, $[x] = \{s \in S \mid s \sim x\}$.

Note. The properties of an equivalence relation on a set S allow us to prove that the equivalence classes partition S. This is accomplished in the next two results.

Lemma 2.58. If $\sim$ is an equivalence relation and $[x] \neq [y]$ then $[x] \cap [y] = \varnothing$.

Theorem 2.59. Let $\sim$ be an equivalence relation on a nonempty set S, and let Π be the family of equivalence classes determined by $\sim$. Then Π is a partition of S. This partition Π is called the partition *induced* by $\sim$.

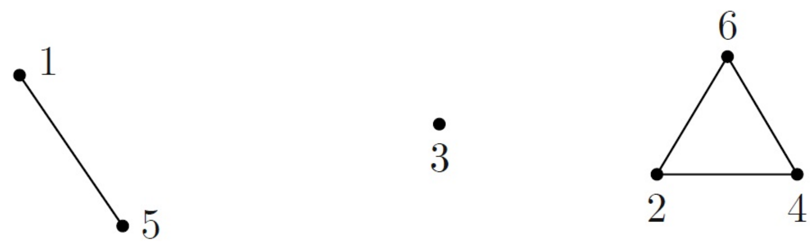
Note/Definition. Consider a finite set S with an equivalence relation $\sim$. We can draw a picture of the equivalence relation by using a point (or *vertex*) to represent each member of S, and connect two vertices with a line segment or an arc (called an *edge*) if the corresponding members of S are related by $\sim$. (By convention, we do not draw an arc from each vertex to itself, even though this would be implies by the reflexive property of $\sim$.) The resulting collection of vertices and edges is a *graph* of the equivalence relation $\sim$. Notice that this idea overlaps with the idea of a "decision tree" (also an example of a graph) which we introduced in Section 2.7. The Power Set.

Example 2.60. Consider the set $S = \{1,2,3,4,5,6\}$ with the equivalence relation

$$\sim = \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6),(1,5),$$

$$(5,1),(2,4),(4,2),(2,6),(6,2),(4,6),(6,4)\}.$$

The graph os this equivalence relation is (from Gerstein, page 92):



Each "cluster" of connected vertices (these are called the *connected components* of the graph) represents an equivalence class, so the associated partition Π of S into equivalence classes is $\Pi = \{\{1,5\},\{3\},\{2,4,6\}\}$.

Note. In fact, the converse of Theorem 2.59 also holds. That is, any partition Π of a set S determines an equivalence relation on set S. Therefore there is a correspondence between the partitions of a set and the equivalence relations on the set. In addition, the blocks of the partition correspond to the equivalence classes of the equivalence relation.

Theorem 2.62. Let Π be a partition of the set S. For $x, y \in S$, define $x \sim y$ to mean that x and y belong to the same block of the partition Π. Then $\sim$ is an equivalence relation on S. This is called the equivalence relation *induced* by partition Π.

Example 2.63. **(a)** In Example 2.49 above we partition $\mathbb{Z}$ using the parity of each integer: $\mathbb{Z} = A \cup B$ where A is the set of even integers and B is the set of odd integers. The equivalence relation $\sim$ induced by this partition is called "congruence modulo 2." This idea, and more general related ones, are explored in Section 6.4. Congruence; Divisibility Tests.

Revised: 2/23/2024