Chapter 3. Functions

Note. In this chapter, we consider functions from one set to another. We will define properties of functions that you will use throughout the rest of you mathematical education and/or career. One could argue that functions are even more fundamental to math than are numbers!

3.1. Definitions and Examples

Note. When you first encounter functions in high school algebra, the main idea is that one "good value" goes into the function and exactly one value comes out of the function. Of course, this initial encounter with functions considers functions with real number inputs and outputs. Sometimes the function is thought of as a machine that produces an output using a (valid) input. See, for example, my online notes for Precalculus 1 (Algebra) on Section 2.1. Functions (notice the figure with the "machine function"). We continue this line of reasoning, but in a more general setting and with a more formal style.

Definition 3.2. Let A and B be sets. A *function f from A to B* is a set of ordered pairs $f \subseteq A \times B$ with the property that from each element x in A there is exactly one element y in B such that $(x, y) \in f$. The statement "f is a function from A to B" is written $f : A \to B$ or $A \xrightarrow{f} B$. Set A is the *domain* of f, denoted $\mathrm{dom}(f)$, and set B is the *codomain* of f. If $(x, y) \in f$ then we write $y = f(x)$; y is the *image* of x and x is a *pre-image* of y.

Example 3.1.A. As usual, in certain settings (when $A = B = \mathbb{R}$, for example) we can define a function with a formula:

$$f = \{(x, y) \mid x \in \mathbb{R}, y \in \mathbb{R}, y = x^2\}.$$

Of course, we abbreviate this as $f(x) = x^2$ for all $x \in \mathbb{R}$. In this case, the domain is $\mathbb{R}$ and the codomain is $\mathbb{R}$ (which is explicit in the definition of f as a set of ordered pairs, but is not explicit in the formula definition). The image of 3 is 9, and the image of -3 is also 9. A pre-image of 9 is 3, and another pre-image of 9 is -3 (notice that we have carefully avoided the square root function here).

Note. Some other terminology associated with function $y = f(x)$ is that f *transforms* or *takes* x to y, denoted $x \overset{f}{\mapsto} y$ or $x \mapsto y$ (with these last symbols read "maps to"). We may say that f *acts on* or *operates on* the members of its domain. Functions are also called *mapping*, *maps*, or *transformations*. You encountered the term "transformation" in Linear Algebra (MATH 2010) in the setting of linear transformations; see my online notes on Section 2.3. Linear Transformations of Euclidean Spaces and Section 3.4. Linear Transformations (of abstract vector spaces). In particular, you saw that a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ can be represented by an $m \times n$ matrix (see Corollary 2.3.A, "Standard Matrix Representation of Linear Transformations" in the online notes for Section 2.3). This is where the importance of the domain and the codomain appears. The domain of linear transformation T tells you what goes into the transformation. The codomain doesn't tell you what comes out (that is, it may not be the range of the transformation), but it tells you the *kind* of thing that comes out: a vector in $\mathbb{R}^m$. This will be something you commonly encounter in math, where functions map one kind of thing to another kind of thing (the "things" being elements of sets in the broadest sense of a "function").

Example 3.3. (a) Suppose $S \subseteq T$. Define function $i : S \to T$ by $i(x) = x$ for all $x \in S$. This is the *inclusion mapping*. Notice that it is not the identity, since it maps the elements of S into a (presumably) *different* set T. If $S = T$ then i is the identity mapping on S, often denoted i_S and also sometimes denoted 1_S.
(e) Let S be the set of all propositions. That is, S is the set of all truth-valued statements (assuming this is a valid set). Define the *truth-valued function* $v : S \to \{\mathrm{T}, \mathrm{F}\}$ as

$$v(P) = \begin{cases} \mathrm{T} & \text{if P is a true proposition} \\ \mathrm{F} & \text{if P is a false proposition.} \end{cases}$$

Based on what we know about logical connectives from Section 1.2. Logical Connectives and Truth Tables, we can state some properties of v. For example, if $v(P) = \mathrm{T}$ and $v(Q) = \mathrm{F}$, then $v(P \wedge A) = \mathrm{F}$ and $v(P \vee Q) = \mathrm{T}$.

Note. A concern in defining a function is that it be "well-defined." This means that the rule of assigning values of $f(x)$ for a given x is unambiguous and consistent. For example, if we define a function f that maps the sets $N_0 = \{0, 3, 6, 9, \ldots\}$, $N_1 = \{1, 4, 7, 10, \ldots\}$, and $N_2 = \{2, 5, 8, 11, \ldots\}$ to $\{0, 1, 2\}$ where we define $f(N_i) = f(n) = n - 3\lfloor n/3 \rfloor$ where $n \in N_i$ for $i \in \{0, 1, 2\}$ (here, $\lfloor x \rfloor$ is the greatest integer less than or equal to x; see Example 2.5.4 in my online notes for Calculus 1 on Section 2.5. Continuity and see Example 3.7 where this is called the *greatest integer function*, denoted $[x]$). Now we have defined f on a set by defining f on an *element* (or a "representative") of the set. To check that f is well-defined, we need to insure that every element of set N_i is mapped to the same element of the codomain $\{0, 1, 2\}$. Notice that $n - 3\lfloor n/3 \rfloor$ is the remainder when the remainder when n is divided by 3 (convince yourself of this). Notice that the remainder of an element of N_i when divided by 3 is i, regardless of which element of N_i we choose. Therefore, f is well-defined. In fact, $f(N_i) = f(n) = i$ where $n \in N_i$ for $i \in \{0, 1, 2\}$.

Note. Since a function is a set of ordered pairs, to say that functions f and g are *equal functions*, denoted $f = g$, is to say $\mathrm{dom}(f) = \mathrm{dom}(g)$ and $f(x) = g(x)$ $\forall x \in \mathrm{dom}(f)$.

Example 3.5. Consider the functions f, g, h defined by

$$\begin{aligned} f(x) &= x^2 - 1 \ \forall x \in \mathbb{Z} \\ g(x) &= x^2 - 1 \ \forall x \in \mathbb{R} \\ h(x) &= (x+1)(x-1) \ \forall x \in \mathbb{R}. \end{aligned}$$

Then $g = h$, but $f \neq g$ because $\mathrm{dom}(f) \neq \mathrm{dom}(g)$. But functions f and g agree on the common values where they are defined. We call f the *restriction* of g to $\mathbb{Z}$, denoted $f = g|_{\mathbb{Z}}$.

Definition 3.6. Let f and g be functions. Suppose $\mathrm{dom}(f) = A \subseteq \mathrm{dom}(g) = B$, and suppose that $f(x) = g(x)$ for all $x \in A$. Then f is the *restriction of g to A*, denoted $f = g|_A$, and g is the *extension of f to B*.