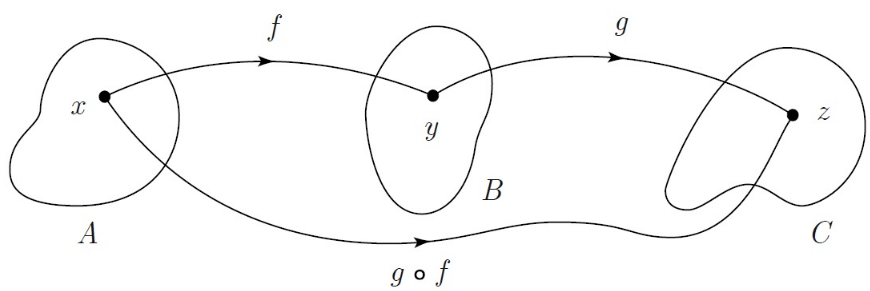
3.3. Composition of Functions

Note. In this section, after defining compositions of functions, we give some properties of compositions including the behavior of compositions of surjections, injections, and bijections. We show that function composition is associative and illustrate some surprising results concerning functions between infinite sets.

Definition. Consider functions $f : A \to B$ and $g : B \to C$. The function $g \circ f : A \to C$ defined as $(g \circ f)(x) = g(f(x))\ \forall x \in A$ is the *composition* of f and g.

Note. We can diagrammatically represent the composition of f and g as:



Example 3.21. (b) Consider the functions f and g from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R} \times \mathbb{R}$ defined by $f(x, y) = (x, 0)$ and $g(x, y) = (0, y)$. These functions are called *projections*, since f projects each point onto the x-axis and g projects each point onto the y-axis. The composite functions $g \circ f$ and $f \circ g$ are

$$(g \circ f)(x, y) = g(f(x, y)) = g(x, 0) = (0, 0) \text{ and}$$

$$(f \circ g)(x, y) = f(g(x, y)) = f(0, y) = (0, 0).$$

In this case, $g \circ f = f \circ g$, though this is rare. In Example 3.21(a), it is shown that for $f(x) = x + 3$ and $g(x) = x^2$ (where both map $\mathbb{R} \to \mathbb{R}$) that $g \circ f \neq f \circ g$ (see page 132).

Example. When first encountering compositions of functions in an algebra class, you might want to take a "complicated function" and decompose it into a composition of several simpler functions (you have to recognize functions as compositions when using the Chain Rule in calculus, also). See my online notes for Precalculus 1 Algebra (MATH 1710) with worked problems from Section 5.1. Composite Functions (notice Exercise 48). Consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = (x^2 + \cos x^2)^3$. We can write these as a composition of $g(x) = x^2$, $h(x) = x + \cos x$, and $k(x) = x^3$, as:

$$f(x) = (x^2 + \cos x^2)^3 = (g(x) + \cos g(x))^3 = (h(g(x))^3 = k(h(g(x))) = (k \circ h \circ g)(x).$$

Note. In Introduction to Modern Algebra (MATH 4127/5127) you will consider a structure called a "group." Elements of a group interact with each other two at a time (through a "binary operation") to produce another element of the group. This interaction is required to satisfy the associativity property. The next theorem shows that function composition is associative. This will allow you to sometimes create groups using functions, where the operation on two functions will be that of composition. In Introduction to Modern Algebra, Section I.2. Binary Operations, the next theorem addressed (it is Theorem 2.13 in Section I.2, though it is not present in the online notes).

Theorem 3.23. Associative Law of Function Composition.

Given functions $f : A \to B$, $g : B \to C$, and $h : C \to D$, then $h \circ (g \circ f) = (h \circ g) \circ f$.

Note. Theorem 3.23 allows us to unambiguously write a composition of three functions as $h \circ g \circ f = h \circ (g \circ f) = (h \circ g) \circ f$. The next theorem is central and considers compositions of injections, surjections, and bijections.

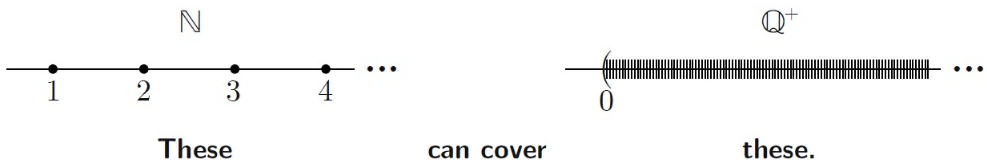
Theorem 3.24. Let $f : A \to B$ and $g : B \to C$ be functions.

(a) If f and g are injections then $g \circ f : A \to C$ is an injection.

(b) If f and g are surjections then so is $g \circ f$.

(c) If f and g are bijections then so is $g \circ f$.

Example 3.25. Let $\mathbb{Q}^+$ denote the set of positive rational numbers. We gave a bijection $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$. Define function $g : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}^+$ as $g((a, b)) = a/b$. Notice that g is surjective, since for any $r = p/q \in \mathbb{Q}^+$ (where p and q are positive integers) we have $(p, q) \in \mathbb{N}\times\mathbb{N}$ and $g((p, q)) = p/q = r$. Of course g is not injective since, for example, $g((2, 3)) = g((4, 6)) = 2/3$ (and $(2, 3) \neq (4, 6)$). By Theorem 3.24(b), we have $g \circ f : \mathbb{N} \to \mathbb{Q}^+$ is surjective. This is surprising when we try to visualize it (see the figure below from Gerstein's page 135). On the real number line, the natural numbers are evenly space one after the other. The rationals are distributed in a very different way. There is no "first" (or least) element of $\mathbb{Q}^+$. The rationals are not spaced out evenly, but instead between any two rationals there are an infinite number of other rationals! It may seem strange that you could take the natural numbers and "cover" the positive rational numbers in this way (and with function $g \circ f$, we even cover each rational an infinite number of times!). It turns out that there is a bijection from $\mathbb{N}$ to $\mathbb{Q}^+$ (and there is also a bijection from $\mathbb{N}$ to $\mathbb{Q}$). Read into this that infinite sets can behave in some surprising ways.



Definition. Suppose $f : A \to B$ is a bijection. Define the function $f^{-1} : B \to A$ where $f^{-1} = \{(b, a) \in B \times A \mid (a, b) \in f\}$. Function f^{-1} is the *inverse* of bijection f.

Note 3.3.A. For bijection f and its inverse f^{-1}, we have the following equivalent statements concerning function values:

$$f^{-1}(b) = a \Leftrightarrow (b, a) \in f^{-1} \Leftrightarrow (a, b) \in f \Leftrightarrow f(a) = b.$$

By Note 3.2.C (which describes injections and surjections in terms of properties of the ordered pairs of the function), we have that f^{-1} is also a bijection.

Theorem 3.27. Let $f : A \to B$ and $g : B \to A$. Then the following two statements are equivalent:

(a) f is a bijection and $g = f^{-1}$.

(b) $g \circ f = i_A$ and $f \circ g = i_B$.

Note. You are likely familiar with finding the inverse of a given algebraic function from your precalculus days. Gerstein states (see page 137):

> "If a bijection f is given by $y = f(x)$, then the formula for f^{-1} is obtained by solving for x in terms of y. If so desired, the variables can then be relabeled by using the letter y in place of x, and x in place of y."

The interchanging of x and y has a geometric implication for the graphs of $y = f(x)$ and $y = f^{-1}(x)$ (namely, they are symmetric with respect to the line $y = x$ in the Cartesian plane when f is a real-valued function of a real variable; a bijection between its domain and range), and we have that "whatever f does, f^{-1} undoes." Additional elementary details are given in my online notes for Precalculus 1 Algebra (MATH 1710) on Section 5.2. One-to-One Functions; Inverse Functions. The proof of the next result is left as an exercise.

Corollary 3.28. Cancellation Laws.

Let f be a bijection. Then:

$$f \circ g = f \circ h \Rightarrow g = h \text{ (left cancellation)}$$

$$r \circ f = s \circ f \Rightarrow r = s \text{ (right cancellation)}$$

Theorem 3.29. Let $f : A \to B$ and $g : B \to C$ be bijections. Then we have $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Revised: 2/2/2022