Chapter 4. Finite and Infinite Sets

4.1. Cardinality; Fundamental Counting Principles

Note. In this section, we start the conversation by consider the "Hilbert hotel" and some of the surprising behavior of infinite sets. We define what it means for two sets to be of the same cardinality (though we don't define a "cardinal number"). We show that all intervals of real numbers are of the same cardinality, we define finite sets and infinite sets, we prove some unsurprising results on the cardinalities of finite sets (including the Pigeonhole Principle), and prove several counting principles related to finite sets.

Note. To start the discussion of infinite sets, Gerstein tells a story about a hotel that has (countably) infinitely many rooms labeled 1, 2, ..., where each room is occupied. If another person shows up and wants a room, then the current occupants can move to different rooms in such a way as to free up a room. The person in room n can simply move to room $n+1$ for each $n \in \mathbb{N}$. Then room 1 is available and the new person can stay in that room. If k new people show up, then they can be accommodated by moving the person in room n to room $n+k$. In fact, if a (countably) infinite number of people show up looking for a room, then they can also be accommodated. Simply move the person in room n to room $2n$ for $n \in \mathbb{N}$. This frees up all of the odd numbered rooms and the infinite number of new people can then occupy those rooms. In this story, the hotel is often called the "Hilbert hotel" after the famous 19th and 20th century mathematician David Hilbert (January 23, 182–February 14, 1943).

Definition 4.1. Sets A and B have the *same cardinality* or the *same cardinal number* if there is a bijection $f : A \to B$. We then say that A and B are *equipotent* sets, denoted $A \approx B$. We may also say that sets A and B are *in one-to-one correspondence*. If A and B are not equipotent, then we write $A \not\approx B$.

Note. The next result gives some (unsurprising) properties of equipotent sets.

Theorem 4.2. Let A, B, C be sets. Then

(a) $A \approx A$,

(b) $A \approx B$ implies $B \approx A$, and

(c) $A \approx B$ and $B \approx C$ implies $A \approx C$.

Consequently, if S is any collection of sets, then equipotence is an equivalenece relation on S.

Example 4.4. Let X be a set with ten elements, let S be the set of all seven-element subsets of X, and let T be the set of all three-element subsets of X. Then $S \approx T$.

Example 4.5. If $a, b \in \mathbb{R}$ with $a < b$, consider the open interval $I_a^b = \{x \in \mathbb{R} \mid a < x < b\}$. Define $f : I_a^b \to I_0^1$ by $x \overset{f}{\mapsto} \dfrac{x-a}{b-a}$. Notice that the graph of f in the Cartesian plane is a line of slope $1/(b-a) > 0$ that maps a to 0 and maps b to 1. So $x_1 < x_2$ implies that $f(x_1) < f(x_2)$, and hence f is injective (one to one). Also, f is surjective (onto) since for any $t \in I_0^1$ we have for $x = a + t(b-a) \in I_a^b$ that $f(x) = t$. That is, f is a bijection and so $I_a^b \approx I_0^1$. Since a and b are arbitrary real numbers (with $a < b$), then we see that any bounded interval of real numbers is equipotent with the interval I_0^1 and, by Theorem 4.2(c), any two bounded intervals of real numbers are equipotent. Recall that the mapping $x \mapsto \tan x$ is a bijection mapping $I_{-\pi/2}^{\pi/2}$ to $\mathbb{R}$ (consider the graph of $y = \tan x$), so that $I_{-\pi/2}^{\pi/2} \approx \mathbb{R}$. As Gerstein puts it (page 144):

> "Every open interval, no matter how short, is in one-to-one correspondence with the set $\mathbb{R}$ of real numbers. Therefore any two open intervals have the same cardinality (even if one interval is properly contained in the other)."

Definition 4.7. A set S is *finite* if $S = \varnothing$ or $S \approx \mathbb{N}_n = \{x \in \mathbb{N} \mid 1 \le x \le n\}$ for some $n \ge 1$ (where n is some given element of $\mathbb{N}$). We denote this as $\#\varnothing = 0$ and $\#S = n$, respectively. We read $\#S$ as "the number of elements in S," provided S is a finite set). A set is *infinite* if it is not finite. To *count* a finite set S is to establish a bijection $f : \mathbb{N}_n \to S$ for some $n \in \mathbb{N}$ or to recognize S is empty.

Note. We now state and prove several counting results which are intuitively clear but sometimes a challenge to prove.

Theorem 4.8. Let n and m be nonnegative integers with $n > m$.

(a) There is no injection from $\mathbb{N}_n$ to $\mathbb{N}_m$, and hence $\mathbb{N}_n \not\approx \mathbb{N}_m$.

(b) If A is a set and $\#A = n$, then $\#A \neq m$.

Corollary 4.9. The Pigeonhole Principle.
Let A and B be nonempty finite sets, with $\#A > \#B$. Then there is no injection from A to B. Thus for any function $A \to B$, some element in B has at least two preimages.

Note. Gerstein describes the pigeonhole principle intuitively as (page 146): "'If n pigeons fly into m pigeonholes and $n > m$, then some pigeonhole receives at least two pigeons." A "pigeonhole" also represents a collection of boxes into which into which something is put (possibly pigeons, but more commonly mail). If the number of pieces of mail (say) exceeds the number of boxes, then some box must get at least two pieces of mail.

Example 4.10(b). We claim that in any set of eleven integers, there are two whose difference is divisible by 10. Let set A be the set of 11 integers so that $\#A = 11$. Let $B = \{0, 1, 2, \ldots, 9\}$ so that $\#B = 10$. Define $f : A \to B$ where f maps $a \in A$ to the its right-hand digit. Since $11 = \#A > \#B = 10$, then by the Pigeonhole Principle f is not an injection. So it must be that some two elements $a_1, a_2 \in A$ are mapped by f to the same element of B, $f(a_1) = f(a_2) = b$. But then the right-most digit of $a_1 - a_2$ is 0 and so $a_1 - a_2$ is divisible by 10. That is, a_1 and a_2 are two integers in set A whose difference is divisible by 10.

Theorem 4.11. Every subset of $\mathbb{N}_n$ is finite, and if $A \subset \mathbb{N}_n$ (that is, A is a proper subset of $\mathbb{N}_n$, $A \subsetneq \mathbb{N}_n$) then $\#A = m$ for some $m < n$.

Theorem 4.12.

(a) Every subset of a finite set is finite.

(b) Every set containing an infinite set is infinite.

(c) If $A \subset B$ (that is, $A \subsetneq B$) and B is finite then $\#A < \#B$.

Note. We have not formally shown that an infinite set exists yet. The next result finally resolves this. Of course we never questioned the existence of infinite sets, but this result shows that our approach (in terms of definitions and axioms, though axioms have played a minor role here) is a reasonable one.

Theorem 4.13. The set $\mathbb{N}$ of natural numbers is infinite.

Note. We now consider several results which relate to counting finite sets. You may have seen some of these results in Foundations of Probability and Statistics-Calculus (MATH 2050).

Theorem 4.14. Addition Rule. If A and B are disjoint finite sets, then $A \cup B$ is finite and $\#(A \cup B) = \#A + \#B$.

Note. Recall that sets $A_1, A_2, \ldots, A_m$ are *pairwise disjoint* if $A_i \cap A_j = \varnothing$ whenever $i \neq j$ (see Exercise 2.6.4). The next result is a generalization of the Addition Rule and follows by induction from Theorem 4.14.

Corollary 4.15. If $A_1, A_2, \ldots, A_m$ are pairwise disjoint finite sets, then $\cup_{i=1}^m A_i$ is finite and

$$\#\left(\bigcup_{i=1}^m A_i\right) = \#A_1 + \#A_2 + \cdots + \#A_m.$$

Note. The next result considers $\#(A \cup B)$ for finite A and B, but where A and B may not be disjoint. This is a special case of the *Inclusion Exclusion Principle* which arises in elementary probability theory. See my online notes for Mathematical Statistics 1 (STAT 4047/5047) on Section 1.3. The Probability Set Function; notice Theorem 1.3.A and Theorem 1.3.B. My online notes for Intermediate Probability and Statistics (not an official ETSU class) on Section 1.10. The Probability of a Union of Events includes an inductive proof of the general Inclusion Exclusion Principle (these statistics notes deal with the principle in terms of probabilities instead of cardinalities, but the technique of proof is easily translated to our setting). The Inclusion Exclusion Principle for three sets is addressed in Exercise 4.1.9 and the general case is considered in our Section 6.6. The Inclusion-Exclusion Principle and Euler's Function.

Corollary 4.16. If A and B are finite sets (not necessarily disjoint), then $A \cup B$ is finite and

$$\#(A \cup B) = \#A + \#B - \#(A \cap B).$$

Note. We now turn our attention from unions of sets to Cartesian products of two sets.

Theorem 4.17. if $\#A = m$ and $\#B = n$, then $\#(A \times B) = mn$.

Corollary 4.18. Let $A = \{a_1, a_2, \ldots, a_m\}$, and for each i satisfying $1 \leq i \leq m$, let B_i be a set with $\#B_i = n$. Then $\#(\cup_{i=1}^m (\{a_i\} \times B_i)) = mn$.

Note. Gerstein gives the following "intuitive interpretation" of Corollary 4.18 (see page 151):

> "Suppose two choices are to be made in succession. If there are m possibilities for the first choice and, once the first choice has been made, n possibilities for the secondindependent of the outcome of the first choicethen there are mn possibilities for the ordered pair of choices.

In introductory probability, this idea is called the *Multiplication Rule*; see my online notes for Mathematical Statistics 1 (STAT 4047/5047) on Section 1.3. The Probability Set Function (notice Rule 1) and my online notes for Intermediate Probability and Statistics (not an official ETSU course) on Section 1.7. Counting Methods (see Theorem 1.7.2.).

Note. Recall that we defined the Cartesian product of *two* sets in Section 2.8. Ordered Pairs and Cartesian Products; see Definition 2.43. We now define the Cartesian product of any finite number of sets.

Definition 4.20. Let $A_1, A_2, \ldots, A_n$ be sets. The *Cartesian product* $A_1 \times A_2 \times \cdots \times A_n$ is the set of all n-tuples $(a_1, a_2, \ldots, a_n)$ such that $a_i \in A_i$ for $1 \leq i \leq n$.

Note. We now generalize Theorem 4.17 to Cartesian products of more than two sets. Of course, the Multiplication Rule can be applied to this setting as well.

Theorem 4.21. Suppose $A_1, A_2, \ldots, A_n$ are finite sets, with $\#A_i = m_i$ for $1 \leq i \leq n$. Then the Cartesian product $A_1 \times A_2 \times \cdots \times A_n$ is also finite, and $\#(A_1 \times A_2 \times \cdots \times A_n) = m_1 m_2 \cdots m_n$.

Example 4.23. A state motor vehicle bureau has decided that all license plates must have exactly seven characters. They agree that the characters will be chosen from the English alphabet A, B, ..., Z or from the numerals 0, 1, ..., 9, but they are still discussing whether the characters should satisfy some further conditions. In each of the following cases determine the number of licenses that satisfy the given condition.

(a) Each character can be any letter or numeral.

Solution. Combining the 26 letters and 10 numerals, there are 36 possible characters. With $n = 7$, $A_1 = A_2 = \cdots = A_7$ equal to the set of 36 characters, so that $m_1 = m_2 = \cdots = m_7 = 36$, and treating a license plate as a 7-tuple, we have that the possible number of license plates is $m_1 \cdot m_2 \cdot \cdots \cdot m_7 = 36^7$, by Theorem 4.21.

(b) The first three characters are letters and the last four are numerals.

Solution. Again with $n = 7$, but this time with $A_1 = A_2 = A_3$ as the set of 26 letters, and $A_4 = A_5 = A_6 = A_7$ as the set of 10 numerals, we now have by Theorem 4.21 that the possible number of license plates is $m_1 \cdot m_2 \cdot \cdots \cdot m_7 = (26^3)(10^4)$.

(c) Three consecutive characters (not necessarily the first three) are letters and the others are numerals.

Solution. We use the Multiplicative Rule more "hands on" here. First, we decide how many ways there are to choose the location of the three consecutive letters. The first of these letters can appear in the first, second, third, fourth, or fifth position, so there are 5 possible choices for the locations of the letters. As in (b), there are then $(26)^3$ ways to choose the three consecutive letters, and there are $(10)^4$ ways to choose the four numerals. So the total number of possible licenses is $5(26)^3(10)^4$.

(d) Exactly three characters are letters and the others are numerals.

Solution. Similar to pat (c), we first decide how many ways there are to choose the location of the three letters. This is computed as "7 choose 3," which is

$$\binom{7}{3} = \frac{7!}{3!(7-3)!} = \frac{7!}{3!4!} = \frac{(7)(6)(5)}{(3)(2)(1)} = 35.$$

We will discuss this idea of *combinations* more in Chapter 5, but you were also exposed to this in Calculus 2 (MATH 1920); see my online Calculus 2 notes on Section 10.10. The Binomial Series and Applications of Taylor Series where combinations are used in determining the binomial coefficients. As in part (c), each of these 35 cases then allow $(26)^3(10)^4$ license plates, for a total of $35(26)^3(10)^4$ possible licenses.

(e) Suppose the agreement that each license will have exactly seven characters is abandoned. Instead it is decided that a license must have at most seven characters, and each character can be any letter or numeral. (No car will get a blank license.)

Solution. Since we now can use all 36 characters, there are 36 licenses with one character, and (by repeated application of Theorem 4.21) 36^2 licenses with two characters, 36^3 licenses with three characters, $\ldots$, 36^7 licenses with seven characters. So the total number of licenses is (Corollary 4.15, if you like) $36 + 36^2 + 36^3 + \cdots = 36^7$. $\square$

Revised: 2/4/2022