4.2. Comparing Sets, Finite or Infinite

Note. In this section, we extend some of the cardinality ideas for finite sets (from the previous section) to infinite sets. In particular, we define what it means for one infinite set to have a cardinality less than (and less than or equal to) the cardinality of another infinite set. We start with a result concerning the empty set.

Lemma 4.25. If A is any set, then $\varnothing$ is an injective function from $\varnothing$ to A.

Note. The next result concerning cardinalities of finite sets and the existence of injections and/or bijections is the motivation for how we deal with infinite sets when addressing cardinalities.

Theorem 4.26. Let A and B be finite sets. Then

(a) $\#A \leq \#B \Leftrightarrow$ There is an injection from A to B,

(b) $\#A = \#B \Leftrightarrow A \approx B$, and

(c) $\#A < \#B \Leftrightarrow$ There is an injection but no bijection from A to B.

Note. Since we can just as easily consider injections and bijections between infinite sets as we can between finite sets, then we use Theorem 4.26 to inspire the next definition. Notice that the definition in the case of finite sets is simply a statement of Theorem 4.26 (so we have consistence with previous ideas).

Definition 4.27. Let A and B be sets (finite or infinite). Define

(a) $\#A \leq \#B$ if there is an injection from A to B,

(b) $\#A = \#B$ if $A \approx B$, and

(c) $\#A < \#B$ if there is an injection but no bijection from A to B.

Note. Gerstein states (see page 160): "We have given meaning to the statement $\#A < \#B$ (and the other statements in [Definition] 4.27) without explicitly defining the symbol $\#A$ when A is infinite. (To do so would take us deeper into the foundations of mathematics then we intend to go)." For those interested in going into this in depth, consider my online notes for Introduction to Set Theory (not a formal class at ETSU), especially the chapters on cardinal numbers and Alephs. A "cardinal number" is associated with the cardinality of a set, and sets of the same cardinality are associated with the same cardinal number. However, it is best to use the expression "the number of elements in a set" only when the set if finite! Gerstein (also on page 160) refers to the blanket use of this expression (even for infinite sets) as using the term 'number' "casually."

Note. We would expect that $\leq$ on cardinal numbers would satisfy the same properties as $\leq$ on the natural numbers. Of course we have $\#A \leq \#A$ since the identity mapping is an injection from A to A. We also have

$$\#A \leq \#B \text{ and } \#B \leq \#C \Rightarrow \#A \leq \#C,$$

since $\#A \leq \#B$ means there exists an injection i_1 from A to B and $\#B \leq \#C$

means there exists an injection i_2 from B to C so that (by Theorem 3.24(a)) $i_2 \circ i_1$ is an injection from A to C, so that $\#A \leq \#C$. But to show transitivity for inequalities involving $<$, we need the result we state next. This result was first published in 1887 by Georg Cantor (March 3, 1845-January 6, 1918), but without proof. Ernst Schröder (November 25, 1841–June 16, 1902) announced a proof in 1896 and independently Felix Bernstein (February 24, 1878–December 3, 1956), at age 19, gave a proof. For this reason, the result is known variously as the Schröder-Bernstein Theorem or the Cantor-Schröder-Bernstein Theorem. This history is from the Schröder-Bernstein Theorem Wikipedia page. We omit the proof but, like Gerstein, metion that a proof can be found in Paul Halmos' *Naive Set Theory* (Princeton: D. Van Nostrand Company, 1960; NY: Springer-Verlag, 1974; Dover Publications, 2017). See my online notes for Naive Set Theory (see Section 22).

Theorem 4.28. Schröder-Bernstein Theorem.
Let A and B be sets. If $\#A \leq \#B$ and $\#B \leq \#A$ then $A \approx B$.

Example 4.29. We now show that the intervals $(0,1)$ and $[0,1]$ are equipotent: $[0,1] \approx (0,1)$. We show the existence of an injection from each of the intervals to the other so that we can conclude $\#(0,1) \leq \#[0,1]$ and $\#[0,1] \leq \#(0,1)$. Then by the Schröder-Bernstein Theorem we can conclude $\#(0,1) = \#[0,1]$. First the inclusion mapping from $(0,1)$ to $[0,1]$ is an injection so that we have $\#(0,1) \leq [0,1]$. Second, in the other direction we map $[0,1]$ to $(0,1)$ by "shrinking" $[0,1]$ uniformly. Consider $f(x) = x/2 + 1/4$. Then f is an injection from $[0,1]$ to $(0,1)$ (in fact, the image of $[0,1]$ under f is $[1/4, 3/4] \subset (0,1)$). Hence $\#[0,1] \leq \#(0,1)$ and the claim now holds. Notice that since we have shown that $[0,1] \approx (0,1)$ then a bijection between the sets exists. Exercise 4.2.4 addresses the construction of a specific bijection.

Theorem 4.30. Let A, B, and C be sets. Then

(a) $\#A < \#B < \#C \Rightarrow \#A < \#C$,

(b) $\#A < \#B \leq \#C \Rightarrow \#A < \#C$, and

(c) $\#A \leq \#B < \#C \Rightarrow \#A < \#C$.

Note. Proof of parts (b) and (c) of Theorem 4.30 are left as exercises. The next result, also due to Georg Cantor, show that some infinite sets have larger cardinalities than others. The corollary to it shows that there is not "largest set" (in terms of cardinality.

Theorem 4.31. Cantor's Theorem (I).
Let S be a set with power set $P(S)$. Then $\#S < \#P(S)$.

Corollary 4.32. $\#\mathbb{N} < \#P(\mathbb{N})\#P(P(\mathbb{N})) < \#P(P(P(\mathbb{N}))) < \cdots$.

Note. Georg Ferdinand L.P. Cantor was born in in Saint Petersburg, Russia in 1845. At age 11 his family moved to Germany. He attended the University of Berlin (he was friends with Hermann Schwarz, of the Schwarz Inequality or the Cauchy-Schwarz Inequality from linear algebra). He attended lectures of Karl Weierstrass and Leopold Kronecker. He finished his dissertation on number theory in 1867. At the encouragement of his colleague, Eduard Heine (of the Heine-Borel Theorem fame) his research turned to analysis. He worked on trig series, and defined irrational numbers in terms of convergent sequences of rational numbers. This definition was referenced by Richard Dedekind in his successful development of the idea of completeness of the real numbers in terms of Dedekind cuts. In 1873, Cantor proved that the rational numbers are countable. He also showed the algebraic numbers (that is, real numbers that are roots of polynomials with integer coefficients) are countable. In December 1973 he proved that the real numbers are *not* countable and published this in 1874 (this is where the idea of a one-to-one correspondence enters in relation to cardinality). He further elaborated on these ideas in an 1878 paper. Between 1879 and 1884 he published six papers on set theory. Cantor's theory of sets was not accepted as widely as he had hoped and it was drawing criticism. In May 1884 Cantor had an attack of depression that lasted a few weeks. It was speculated that his depression was the result of the criticism that he suffered through, but in light of modern ideas of mental illness this is not thought to be a major contributor to his problems. Cantor's last major set theory papers were published in 1895 and 1897 (he had been working on the Continuum Hypothesis, the claim that there is no set of cardinality greater than the cardinality of the natural numbers and less than the cardinality of the real numbers, and he wanted to include a proof of this in his papers, but he could not find one; this claim turn out to be neither true nor false under the accepted axioms of set theory...). Following this time and some personal tragedies, Cantor continued to suffer depression. He took a leave from teaching in the winter semester of 1899-1900, and spent time off-and-on in a sanatorium from 1899 onwards. He formally retired in 1913. He entered a sanatorium in 1917 and died of a heart attack January 6, 1918. This biographical information and the following image are from the MacTutor History of Mathematics Archive biography of Cantor.

Georg Cantor

March 3, 1845–January 6, 1918

We give proofs of some of Cantor's results in the next section.

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