4.3. Countable and Uncountable Sets

Note. In this section we finally define a "countable set" and show several sets to be countable (such as $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{N} \times \mathbb{N}$). We prove Cantor's Theorem (II): The real numbers are not countable.

Definition 4.33. A set S is *countably infinite* if $\mathbb{N} \approx S$; that is, if there is a bijection from $\mathbb{N}$ to S. A set is *countable* (or *denumerable*) if it is finite or countably infinite. A set that is not countable is *uncountable*, *uncountably infinite*, or *nondenumerable*.

Note. The natural numbers $\mathbb{N}$ are a very reasonable model for the idea of "countably infinite." For any countable set, there is a first element (say $s_1 = f(1)$ where $f : \mathbb{N} \to S$), a second element $s_2 = f(2)$, and so forth. With this in mind, the next is maybe not too surprising. The real surprise will come below when we show that there is an uncountable set (namely, $\mathbb{R}$). So, no matter how we attempt to "list" the real numbers, no such list exists.

Example 4.34. The set of integers $\mathbb{Z}$ is countably infinite.

Note. The next result formally justifies the Hilbert hotel story (the version given by Gerstein) in Section 4.1. Cardinality; Fundamental Counting Principles.

Theorem 4.35. If A is finite and B is countable then $A \cup B$ is countable.

Theorem 4.36.

(a) Every subset of a countable set is countable.

(b) Every infinite set contains a countably infinite subset.

Note. Like Gerstein, we "leave the proof [of Theorem 4.36] for a later course in set theory." This is covered in Introduction to Set Theory (not a formal ETSU class). A proof of part (a) of the previous theorem is given in Section 4.3. Countable Sets; see Theorem 4.3.2. A proof of (b) of the previous theorem (which requires the Axiom of Choice) is given in Section 9.1. The Axiom of Choice and Its Equivalence; see Theorem 9.1.4. These notes are currently (spring 2022) in preparation and these two links may not work.

Note. We now give some results concerning results for countable sets that are analogous to previous results shown for finite sets. In the rest of this section when claiming a mapping is a bijection, we will not go into the level of detail as we have previously. . . we "leave this as an exercise," if you like!

Theorem 4.37. If A and B are countable sets then $A \cup B$ is countable.

Note. By the Principle of Mathematical Induction, we have the next corollary to Theorem to 4.37.

Corollary 4.38. If $A_1, A_2, \ldots, A_n$ are countable sets, then $\cup_{i=1}^n A_i$ is countable.

Note. We take advantage of the Fundamental Theorem of Arithmetic in the proof of the next result. Recall that a *prime number* $p > 1$ is an integer having no factorization of the form $p = ab$ where $a > 1$ and $b > 1$ are integers. The Fundamental Theorem of Arithmetic states that every integer $n > 1$ can be uniquely written as a product of prime numbers. We'll see a proof of this in this course in Section 6.3. Divisibility: The Fundamental Theorem of Arithmetic; see Theorem 6.29. We also prove this in Elementary Number Theory (MATH 3120); see my online notes for this on Section 2. Unique Factorization; notice Theorem 2.2.

Theorem 4.39.

(a) $\mathbb{N} \times \mathbb{N}$ is countably infinite.

(b) If A and B are countable then $A \times B$ is countable.

Note. In this course we introduce just on infinite "cardinal number." We use the symbol $\aleph_0$, read "aleph naught," to indicate the cardinality of a countably infinite set. If set A is countably infinite, then we write $\#A = \aleph_0$. The next surprising result claims that $\#\mathbb{Q} = \aleph_0$.

Theorem 4.40. The set of rational numbers $\mathbb{Q}$ is countable.

Note. We use the Cantor diagonalization method again (as we did in Section 3.2. Surjections, Injections, Bijections, Sequences) to prove the counterintuitive result that the real numbers are an infinite set larger than the infinite set $\mathbb{N}$ (that is, $\#\mathbb{R} \neq \aleph_0$). The proof requires a unique decimal representation of a real number. We claim that every real number with a finite decimal representation, has a representation in which the last nonzero decimal is decreased by 1 and then followed by an infinite number of 9's. For example, $0.35 = 0.34999\cdots$ and $1.0 = 0.9999\cdots$. You will see the proof of the next theorem again in Analysis 1 (MATH 4217/5217); see my online Analysis 1 notes on Section 1.3. The Completeness Axiom and notice Theorem 1-20.

Theorem 4.41. Cantor's Theorem (II).
The set of real numbers $\mathbb{R}$ is uncountable.

Note. We could have represented each number in $I = (0, 1)$ in binary (i.e., base 2) so that the only digits are 0 and 1. Then Cantor's diagonalization argument is a bit cleaner; we run along the diagonal in the proof and change 0's to 1's and change 1's to 0's.

Corollary 4.42. The set of irrational numbers is uncountable.

Example 4.43. This example gives a cute geometric result using an argument based on cardinalities of sets. Since $\mathbb{Q}$ is countable by Theorem 4.10, the set $\mathbb{Q} \times \mathbb{Q}$

is countable (this follows by an argument similar to that for Theorem 4.39 for $\mathbb{N} \times \mathbb{N}$). In the Cartesian plane, $\mathbb{Q} \times \mathbb{Q}$ corresponds to the points having rational coordinates. If A and B are distinct points in the xy-plane and not in $\mathbb{Q} \times \mathbb{Q}$, then A and B can be connected by a path that contains no points in $\mathbb{Q} \times \mathbb{Q}$.

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