4.4. More on Infinity

Note. In this very brief section, we give a classification of infinite sets. We do so by showing that a set in infinite if and only if it is equipotent with a proper subset (in Theorem 4.44).

Note. In fact, Richard Dedekind (October 6, 1831–February 12, 1916), in his 1888 essay "Was sind und was sollen die zahlen?" (translated as "What are numbers and what should they be?" and "The Nature and Meaning of Numbers"), takes our Theorem 4.44 as the definition of an infinite set. This means that he has defined an infinite set without appealing to the natural numbers $\mathbb{N}$. This appears in Section V. The Finite and Infinite of "The Nature and Meaning of Numbers." This work is still in print as *Essays on the Theory of Numbers* by Richard Dedekind, translated by W. W. Beman, (Dover Publications, 1963). The original publication of this (by Open Court Publishing, 1901) can also be found online at Archive.Org (accessed 2/7/2022). Charles Sanders Peirce (September 10, 1839–April 19, 1914), an American chemist by training, but philosopher by trade (more specifically, logician with crossover interests in math and psychology) also stated the as the definition of an infinite set. In fact, he did so in in 1881 (before Dedekind) in "On the Logic of Number," *American Journal of Mathematics* **4**, 85-95 (1881). This can be found online on Google Books (accessed 2/7/2022).

Theorem 4.44. Let S be a set. Then S is infinite if and only if $S \approx S'$ for some $S' \subset S$ (that is, for some $S' \subsetneq S$).