6.2. The Integers: Operations and Order

Note. In this section, give a brief discussion of the integers and some of their properties. In particular, we define an order $<$ on $\mathbb{Z}$.

Note 6.2.A. You are familiar with the integers $\mathbb{Z}$ and we do not prove many of the fundamental properties of $\mathbb{Z}$. For example, we know that addition $+$ and multiplication $\cdot$ are associative and commutative operations, and that 0 is the additive identity while 1 is the multiplicative identity. Each integer has an additive inverse, but only 1 and -1 have multiplicative inverses. Addition and multiplication interact in the sense that multiplication distributes over addition:

$$a(b+c) = ab + ac \text{ and } (a+b)c = ac + bc \text{ for all } a, b \in \mathbb{Z}$$

(although given commutivity of multiplication, one of these claims implies the other). That is, as we observed in Section 6.1. Operations, the integers are an example of a commutative ring. We also have the property: for any $a, b \in \mathbb{Z}$ if $ab = 0$ then either $a = 0$ or $b = 0$. That is, there are no *zero divisors* in $\mathbb{Z}$. In the verbiage of modern algebra, $\mathbb{Z}$ is an *integral domain.* These last ideas are explored in Introduction to Modern Algebra (MATH 4127/5127); see my online notes for this class on Section IV.19. Integral Domains.

Note. In what follows, we denote the additive inverse of $a \in \mathbb{Z}$ as "$-a$." Strictly speaking, this is not "negative a" but instead is the "additive inverse of a." So a claim made the lines of $a \cdot (-b) = -ab$ (as given in part (b) of the next theorem) is to be read as "a times the additive inverse of b equals the additive inverse of ab."

The claim that $(-a)(-b) = ab$ (see Exercise 6.2.3(b)) is not that "the product of two negatives is positive" (after all, the integer ab here may not be positive), but that "the product of the additive inverses of a and b equals ab"; notice that we do not yet have a definition of "positive," but we will below.

Theorem 6.9. Let $a, b, c \in \mathbb{Z}$. Then

(a) $0 \cdot a = 0$,

(b) $a \cdot (-b) = -ab$,

(c) Cancellation Law: If $a \neq 0$ and $ab = ac$ then $b = c$.

Note/Definition. We have $\mathbb{N} \subset \mathbb{Z}$, and for every $n \in \mathbb{Z}$ exactly one of the following holds: $n \in \mathbb{N}$, $n = 0$, or $-n \in \mathbb{N}$. This is the *trichotomy law* or the *Law of Trichotomy*. We define the order relation "$<$" on $\mathbb{Z}$ as: $a < b \Leftrightarrow b - n = n$ for some $n \in \mathbb{N}$. This is also denoted $b > a$. This same idea of an ordering is dealt with in Analysis 1 (MATH 4217/5217) when defining the real numbers. See my online notes for Analysis 1 on Section 1.2. The Real Numbers, Ordered Fields; notice Axiom 8/Definition of Ordered Field. We have the following easily demonstrated properties of $<$ (see Exercise 6.2.7 for part of the proof).

Theorem 6.12. Ordering Properties.

(a) For all $a, b \in \mathbb{Z}$, exactly one of the following holds: $a < b$, $a = b$, or $a > b$.

(b) (The Transitive Law) For all $a, b, c \in \mathbb{Z}$, if $a < b$ and $b < c$ then $a < c$.

Definition. The *absolute value function* on $\mathbb{Z}$ is defined piecewise as

$$|n| = \begin{cases} n & \text{if } n \geq 0 \\ -n & \text{if } n < 0. \end{cases}$$

Theorem 6.2.A. Properties of Absolute Value.

(a) $|n| \in \mathbb{N} \cup \{0\}$ for all $n \in \mathbb{Z}$.

(b) $|n| = 0 \Leftrightarrow n = 0$. In particular, if $n \neq 0$ then $|n| \geq 1$.

(c) $|ab| = |a|\,|b|$ for all $a, b \in \mathbb{Z}$.

(d) The Triangle Inequality. $|a + b| \leq |a| + |b|$ for all $a, b \in \mathbb{Z}$.

Note. The proofs of the properties given in Theorem 6.2.A are straightforward, except for the Triangle Inequality. For a proof of the Triangle Inequality in the setting of $\mathbb{R}$, see my Analysis 1 (MATH 4217/5217) notes on Section 1.2. The Real Numbers, Ordered Fields; notice Theorem 1-13(h) and the fact that the proof depends on several other results in this section. This is not the first time you have seen the Triangle Inequality, since it arises in Linear Algebra (MATH 2010); see my online notes for Linear Algebra on Section 1.2. The Norm and Dot Product (see Theorem 1.5. The Triangle Inequality) and Section 3.5. Inner-Product Spaces (see Theorem 3.5.A. The Triangle Inequality).

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