Supplement. Archimedes' *Method*, Part 2

Note. In this note, we give some details of the study of the Archimedes palimpsest and the techniques used in the study. We also present some of the mathematics in the *Method* and consider how it changed some of the previously-held ideas in the history of math (in particular, the understanding of the concept of infinity in the time of the ancient Greeks).

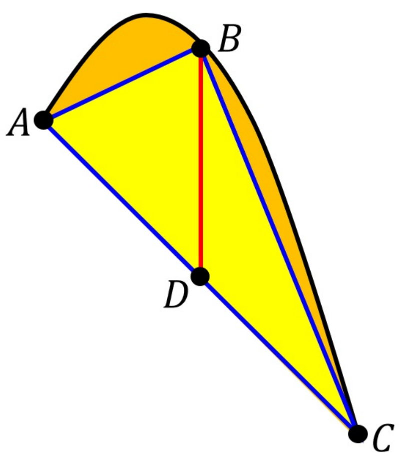
Note AM2.A. We left off in the first part of this supplement, Supplement. Archimedes' *Method*, Part 1, with the palimpsest in the hands of the Walters Art Museum in Baltimore, MD. They took possession of it in 1990 and the work of disbinding it started in April of 2000. The binding was covered with two types of glue: "hide glue" (made from animal skin) and a commercial wood glue. The hide glue was easily removed, but the wood glue was stronger than the parchment and did not lend itself to being dissolved by a solvent. Four of the folios were covered with the forged paintings and the pages had been additionally damaged by the forger who scuffed up the paintings (and the underlying folios) to make them look older. The backs of the folios are also damaged by Blu-tack, a sticky substance probably used in displaying the paintings. After a folios was disbounded, a record of he tears, drops of wax, mold stains, rust, and Blu-tack was made. This note is based on pages 164 and 175 of *The Archimedes Codex*.

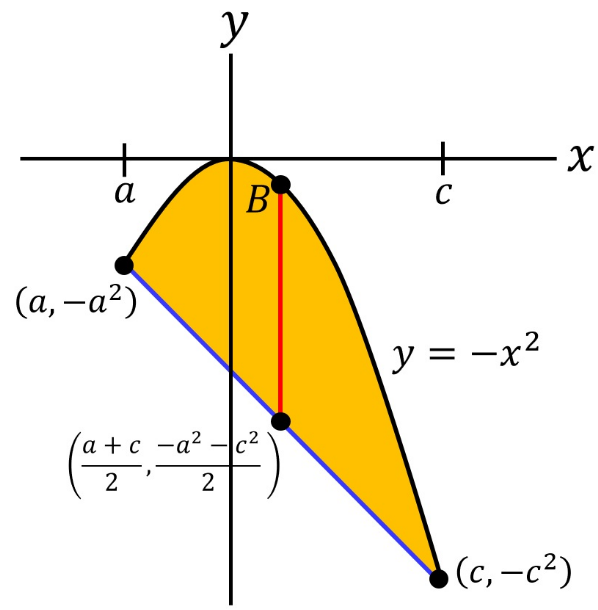
Note AM2.B. There was concern among the history of mathematics community that Johan Heiberg had already extracted all information that one could get from the palimpsest (especially given its current degraded condition); see Notes AM.B and AM.M. The Walters Museum team needed something new to justify the work that they were putting into the project. Some of this came from the first folio of the codex. It contained a whole page of Archimedes' *Floating Bodies* in Greek that Heiberg had overlooked. Heiberg's photographs of the folios of the codex were incomplete, as well. There were whole sections of the *Method* and of *Floating Bodies* that he did not photograph. These parts had not been read, so there was still plenty to learn from the palimpsest. This note is based on page 180 of *The Archimedes Codex*.

Note AM2.C. Many of Archimedes' most famous results involve the area of a region bounded by a "curved line" or the volume of a solid bounded by a curved surface. We first consider the area bounded by a parabola. We now look at Proposition 1 of the *Method*. The proof is a "combination of physics, mathematics, and infinity" (as it is put in *The Archimedes Codex* on page 187). The statement, as given by Thomas Heath in his 1912 supplement "The *Method* of Archimedes" in *The Works of Archimedes* (Cambridge University Press, in print today by Dover Publications), is:

> **Proposition 1.** "Let ABC be a segment of a parabola bounded by the line AC and the parabola ABC, and let D be the middle point of AC. Draw the straight line DBE parallel to the axis of the parabola and join AB, BC. Then shall the segment ABC be $\frac{4}{3}$ of the triangle ABC."

For a version of Proposition 1 in less quaint language, consider the figure below (left). The claim is that the area bounded by the parabolic segment ABC and the line segment AC (the area in orange, if we consider the yellow triangle as obscuring part of this area; see the figure below right) is $4/3$ of the area of triangle $\triangle ABC$ (in yellow).





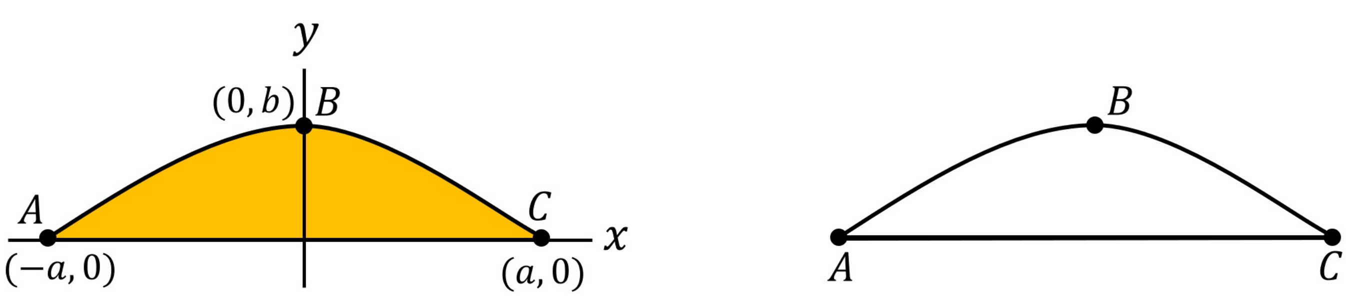
We note that we can easily use calculus and geometry to establish the claim. In the figure above (right), we consider the parabola $y = -x^2$ (which is arbitrary, since all parabolas have the same shape). We let $A = (a, -a^2)$ and $C = (c, c^2)$. The line through points A and C has equation $y = \dfrac{c^2 - a^2}{a - c}x + ac = -(a + c)x + ac$. Recall that the area between continuous functions f and g with $f(x) \geq g(x)$ for $x \in [a, b]$, the area of the region between the curves $y = f(x)$ and $y = g(x)$ from a to b is the integral of $(f - g)$ from a to b (see my online Calculus 1 [MATH 1910] notes on Section 5.6. Substitution and Area Between Curves). Therefore the area in orange is

$$\begin{aligned} \int_a^c (-x^2) - (-(a+c)x + ac)\, dx &= \int_a^c -x^2 + (a+c)x - ac\, dx \\ &= \left(-\frac{1}{3}x^3 + \frac{a+c}{2}x^2 - acx \right)\Bigg|_a^c \end{aligned}$$

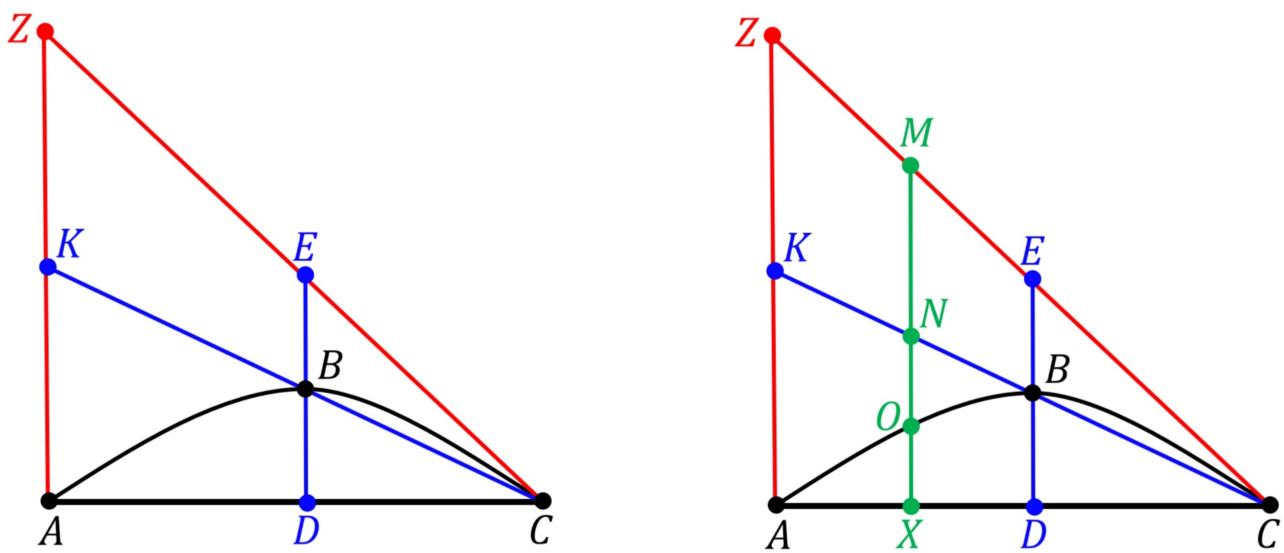
$$\begin{aligned}
&= -\frac{1}{3}(c^3 - a^3) + \frac{a+c}{2}(c^2 - a^2) - ac(c-a) \\
&= \frac{1}{3}a^3 - \frac{1}{3}c^3 + \frac{1}{2}ac^2 - \frac{1}{2}a^3 + \frac{1}{2}c^3 - \frac{1}{2}a^2c \\
&\quad -ac^2 + a^2c \\
&= -\frac{1}{6}a^3 + \frac{1}{6}c^3 - \frac{1}{2}ac^2 + \frac{1}{2}a^2c \\
&= \frac{1}{6}(c^3 - 2ac^2 + 2a^2c - a^3) = \frac{1}{6}(c-a)^3.
\end{aligned}$$

In Exercise AM.1 it is to be shown that the area of triangle $\triangle ABC$ is 3/4 of this value, establishing Proposition 1 with calculus. Next, we give Archimedes' proof of a simplified version of Proposition 1.

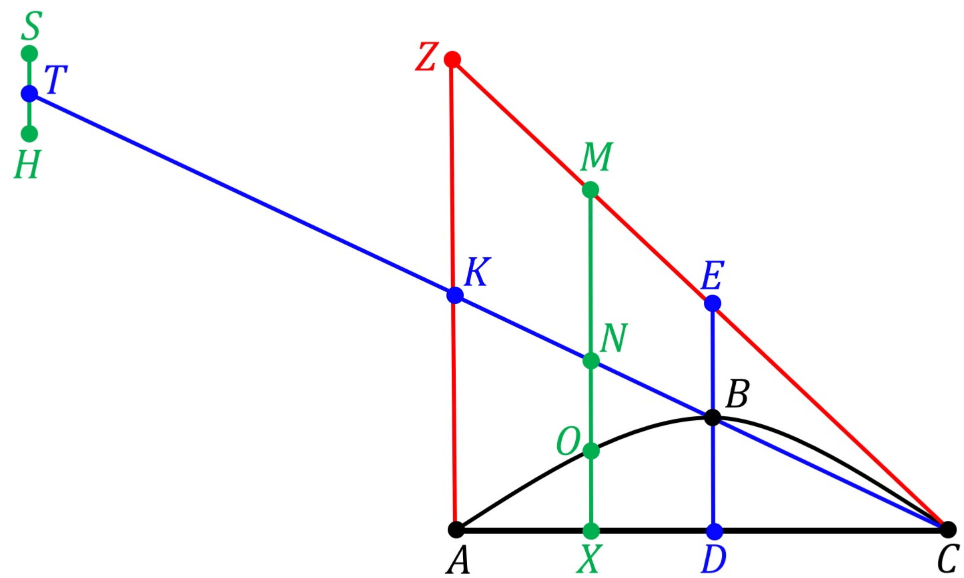
Note AM2.D. We consider the special case of a region bounded by a parabola and a line in which the line is perpendicular to the axis of the parabola (see the figure below).



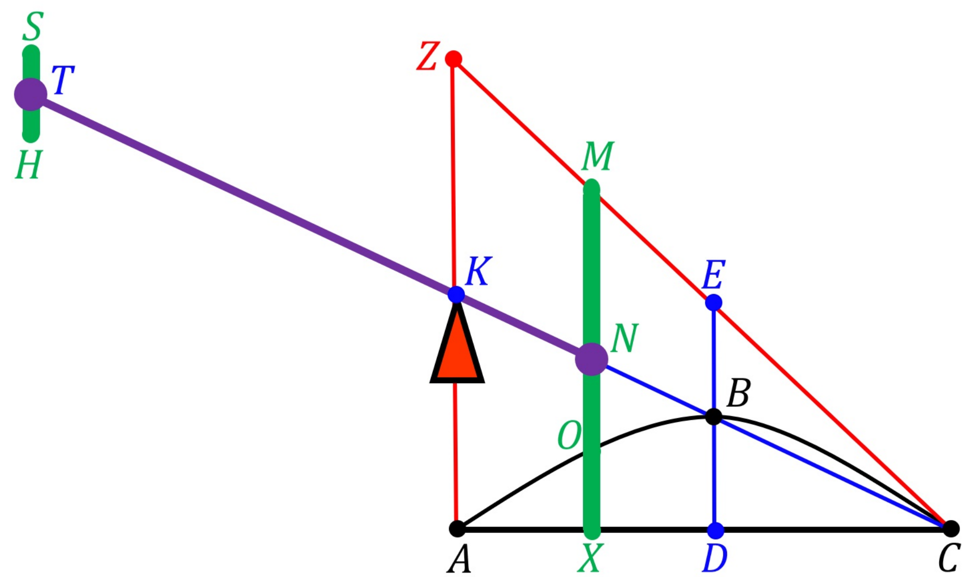
We now give Archimedes' argument, and justify his claims as best we can using results from his era (in Heath's 1912 "The *Method* of Archimedes," there is not justification of every little step, as there is in Euclid's *Elements*). Construct a line tangent to the parabola at point C. This construction would no doubt be known to the ancient Greeks. For such a construction, see my online notes for the history part of Introduction to Modern Geometry (MATH 4157/5157) on Section 3.1. The Parabola (see Note 3.1.C). Construct a perpendicular line to line AC at point A, and label its intersection with the tangent line as Z (this construction is given in Euclid's *Elements*, Book I Proposition 12). These two lines (line segments) are red in the figure below. Introduce the axis of the parabola through point B and label its intersection with line AC as point D and its intersection with line CZ as point E (since the parabola is considered as given, then its axis would be considered as given). Add line CB and let its intersection with line AZ be point K. These two new lines (line segments) are in blue in the figure below (left).



Let X be an arbitrary point between A and C on line segment AC. Add a line perpendicular to AC through point X (Euclid, Proposition I.12), and introduce the points of intersection M, N, and O as given in the figure above (right). These points and line segment MX are in green. Extend line segment CK so that the distance from C to K equals the distance from K to T (that is, K is the midpoint of segment CT). Add line segment SH where the length of SH equals the length of OX (we take T as the midpoint of line segment SH, as in the figure below).



In Apollonius' (circa 262 BCE–circa 190 BCE) *Treatise on Conic Sections*, Book I Proposition 33 gives some properties of parabolas, including the implications that (1) B is the midpoint of segment ED, (2) N is the midpoint of segment MX, and (3) K is the midpoint of segment AZ; these claims are also justified in the *Elements of Conics* by Aristaeus (circa 370 BCE–circa 300 BCE) and Euclid (circa 325 BCE–circa 265 BCE). Apollonius' work is discussed in Section 6.4. Apollonius, and Euclid's lost work on conics is mentioned in Section 5.8. Euclid's Other Works (see Note 5.8.F, where it is stated that Euclid's work is an improvement on an earlier work of Aristaeus). In Archimedes' *Quadrature of the Parabola* (available in Heath's 1897 *The Works of Archimedes*), Proposition 5 implies that $\frac{MX}{OX} = \frac{AC}{AX}$ (where we denote the length of a line segment simply in terms of its endpoints). Since MX and AZ are parallel, then $\frac{AC}{AX} = \frac{KC}{KN}$, and hence $\frac{MX}{OX} = \frac{KC}{KN}$. Since $TK = KC$ then $\frac{MX}{OX} = \frac{TK}{KN}$, and since $SH = OX$ then $\frac{MX}{SH} = \frac{TK}{OX}$ or $MX \times KN = SH \times TK$. Now comes Archimedes' argument based on physics. He interprets lengths MX and SH as "weights" on a balance with fulcrum located at point K (see the figure below; the arms of the balance are in purple).

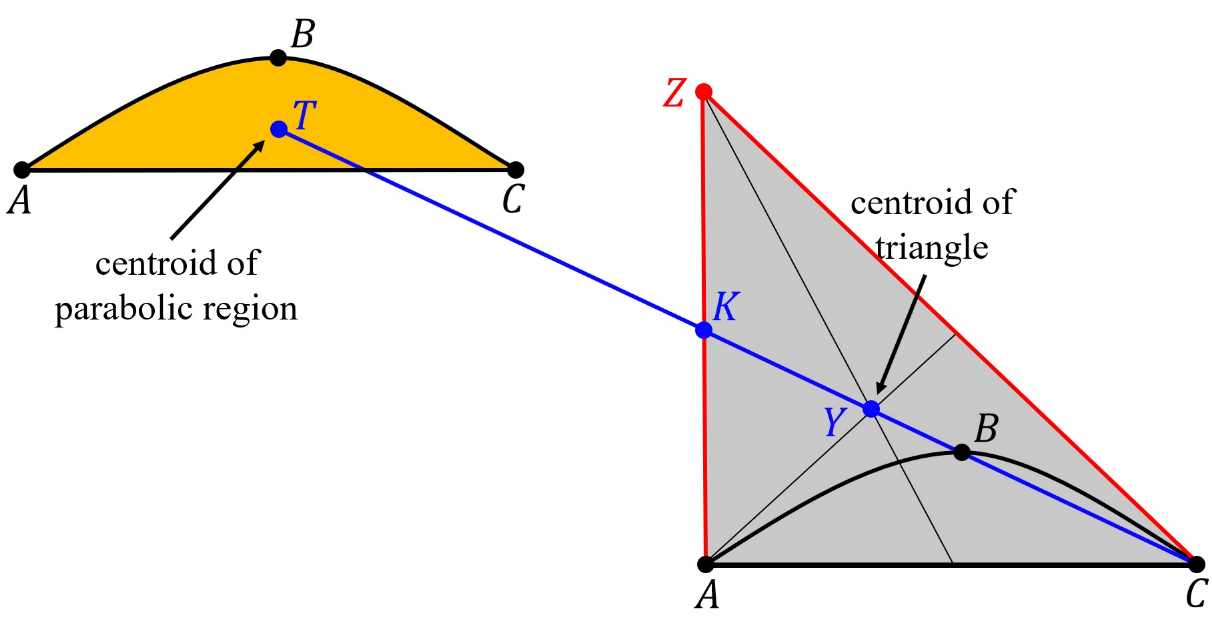


Since we have $MX \times KN = SH \times TK$, then the "weight" MX balances with the "weight" SH when the fulcrum is located at point K. Sometimes called the Law of the Balance, this is established in Archimedes' *On the Equilibrium of Planes*, Book I Propositions 6 and 7 (*On the Equilibrium of Planes* is also in Heath's 1897 *The Works of Archimedes*). We now paraphrase Heath's 1912 English version of the *Method* on this part of the proof (some of the variables have been changed to match the figures above, and the wording is very slightly modified):

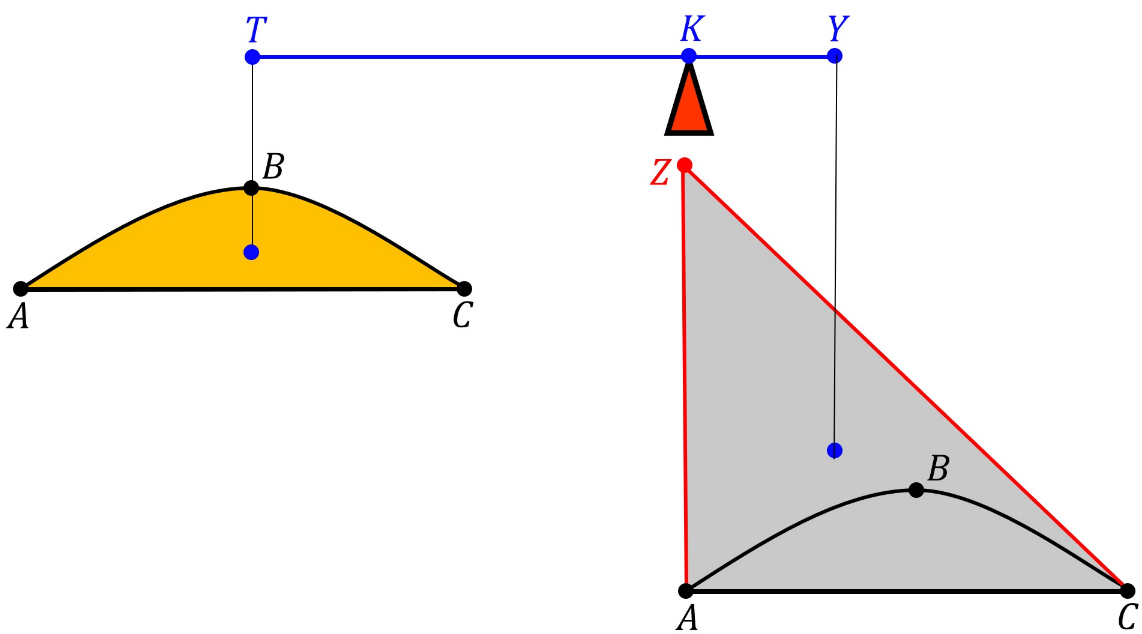
> Therefore K is the center of gravity of the whole system consisting (1) of all straight lines MX intercepted between ZC and AC, and places as they actually are in the figure and (2) of all the straight lines placed at T equal to the straight lines as OX intercepted between the curve and AC.
>
> And, since the triangle $\triangle CZA$ is made up of all the parallel lines line MX,

> and the area bounded under the parabolic segment CBA is made up of all the straight lines like OX within the curve,
>
> it follows that the triangle $\triangle CZA$, placed where it is in the figure, is an equilibrium about K with the area bounded under the parabolic segment CBA placed with its center of gravity at T.

The "it follows" comment is explained in *The Archimedes Codex* as "taking together" all of the line segments which make up triangle $\triangle CZA$ and "taking together" all of the line segment which make up the area bounded under the parabolic segment CBA. Archimedes' argument is that since each pair of slices (i.e., line segments) of the areas balance about point K, then the whole collection of these slices balance.



In the figure above, we place the area bounded under the parabolic segment such that its center of gravity (or centroid) coincides with point T. The centroid of such an area is given in *On the Equilibrium of Planes*, Book II Propositions 4 and 8. We also need to find the centroid of the triangle $\triangle CZA$. This can be found by intersecting two of the three line segments from a vertex of the triangle to a bisector of the opposite side (line segment CK is one such line segment). This is shown in *On the Equilibrium of Planes* Book I Propositions 13 and 14. The centroid of triangle $\triangle CZA$ is labeled Y in the figure above. In Exercise AM.2 it is to be shown that line segment KY is one third of the length of line segment KC; that is, $KY = \dfrac{1}{3}KC$. An alternative view of this in terms of the centroids and a balance is as follows:

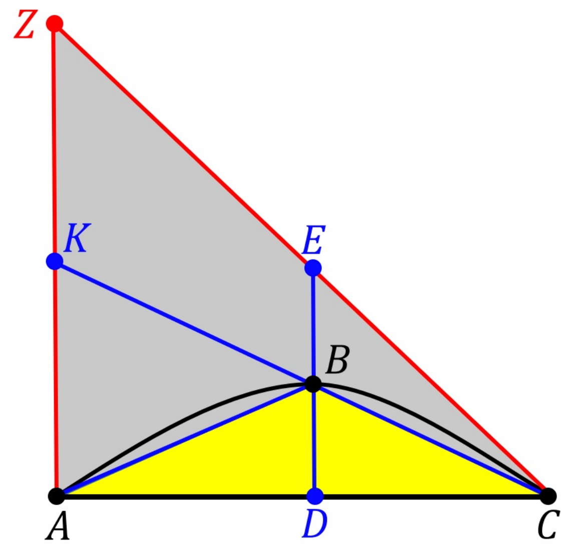


Applying the Law of the Balance we now have that

$$3(\text{area bounded by parabolic arc } ABC) = (\text{area of } \triangle AZC).$$

Archimedes' Proposition 1 claims that the area under the parabola equals the area of triangle $\triangle ABC$. So, we introduce line segment AB and shade triangle ABC yellow in the figure below. Since DE is parallel to AZ, then triangles $\triangle AZC$

and $\triangle DEC$ are similar. Since D is the midpoint of AC, then the area of triangle $\triangle AZC$ is 4 times the area of triangle $\triangle DEC$ (since the base and height of $\triangle AZC$ are twice the base and height, respectively, of $\triangle DEC$).



That is,

$$(\text{area of } \triangle AZC) = 4(\text{area of } \triangle DEC).$$

We now have

$$3(\text{area bounded by parabolic arc } ABC) = (\text{area of } \triangle AZC) = 4(\text{area of } \triangle DEC),$$

or

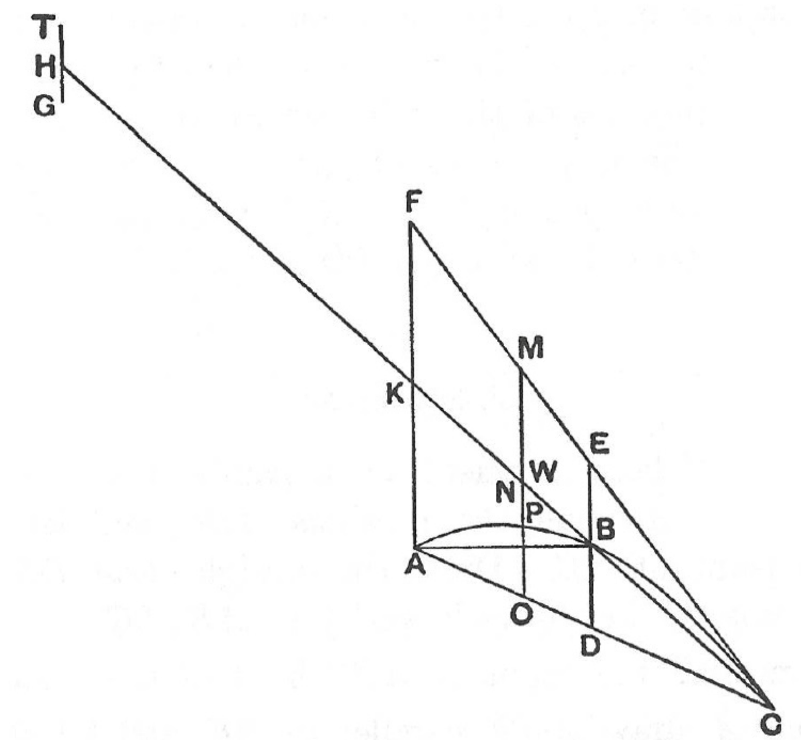
$$(\text{area bounded by parabolic arc } ABC) = \frac{4}{3}(\text{area of } \triangle DEC).$$

Since the base of $\triangle ABC$ is twice the base of $\triangle DEC$, and the height of $\triangle ABC$ is half the height of $\triangle DEC$, then the areas of $\triangle ABC$ and $\triangle DEC$ are equal. Therefore,

$$(\text{area bounded by parabolic arc } ABC) = \frac{4}{3}(\text{area of } \triangle ABC),$$

as claimed. This note is based largely on pages 150–157 of *The Archimedes Codex.*

Note. As mentioned in Note AM2.C, Archimedes considers the more general case where where the parabola is but by an arbitrary line (and not necessarily one which is perpendicular to the axis, as we considered in Note AM2.D). The figure given in Heath's 1912 "The *Method* of Archimedes" (on his page 16), as given below, considers the more general case.

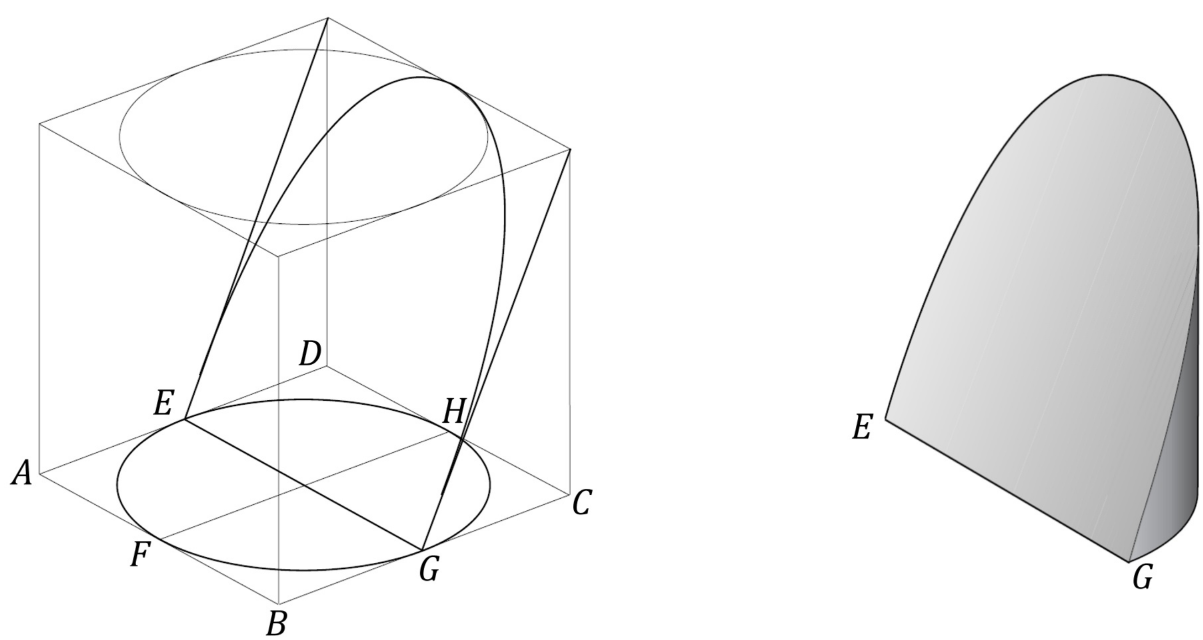


Note AM2.E. The uniqueness of Archimedes' argument given for Proposition 1 in Note AM2.D is twofold. First, he considered geometric objects as if they were physical and had "weight." It is stated on page 155 of *The Archimedes Codex*: "...no one ever did this prior to Archimedes. Just as he invented the mathematical treatment of physics, he has also invented the physical treatment of pure mathematics." Second, he has touched on the topic of integration, some 1900 years before Newton and Leibniz. He has avoided the idea of summations by generalizing from the behavior of cross sections of the areas to the behavior of the areas themselves. You will recall that in Calculus 1 (MATH 1910), we introduce Riemann sums (named after Bernhard Riemann, September 17, 1826–July 20, 1866) and then take limits of these Riemann sums. See my online Calculus 1 notes on Section 5.2. Sigma Notation and Limits of Finite Sums and Section 5.3. The Definite Integral. A consequence of the modern approach is that integrals are *not* sums! Integrals are *limits* of sums. So it is good that Archimedes didn't simply try to sum up cross sections! Of course it isn't a finite sum... it's not even an infinite sum (i.e., a series). It is more complicated, and requires a more subtle approach. This is related (at least indirectly) to the fact that there are different "levels" of infinity. As you see in Mathematical Reasoning (MATH 3000), some infinities are bigger than others! See my online notes for Mathematical Reasoning on Section 4.3. Countable and Uncountable Sets. The cross sections of the areas do not form a countable set. Archimedes avoid summing all together in proving Proposition 1 by "taking together" all of the cross sections. The exploration of the Archimedes palimpsest would reveal that his understanding of this process was deeper than the proof of Proposition 1 reveals. This would lead to a reinterpretation of Archimedes' contributions and a revision of the history of mathematics.

Note AM2.F. In Proposition 1, Archimedes considered an area bounded by a "curved line." His Proposition 14 involves finding the volume of a solid bounded, in part, by a cylinder. In Heath's 1912 "The *Method* of Archimedes," Proposition 14 is stated as follows:

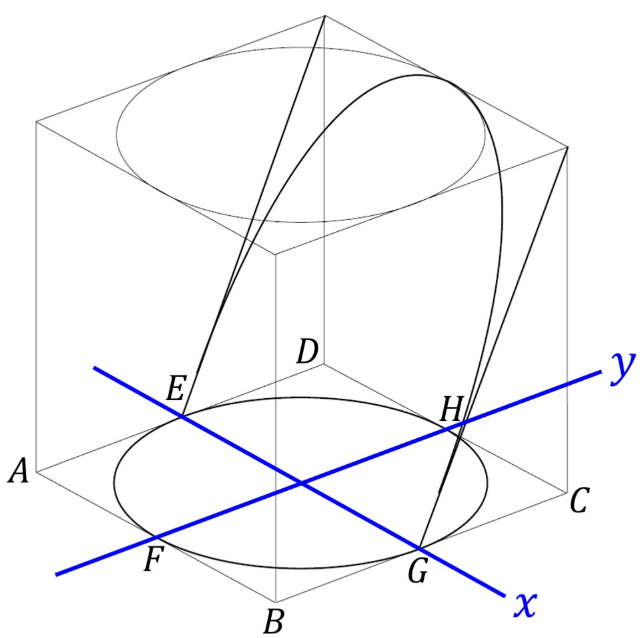
> **Proposition 14.** "Let there be a right prism with square base (and a cylinder inscribed therein having its base in the square $ABCD$ and touching its sides at $EFGH$; let the cylinder be cut be a plane through EG and the side corresponding to CD in the square face opposite to $ABCD$). This plane cuts off from the prism a prism, and from the cylinder a portion of it. It can be proved that the portion of the cylinder cut off by the plane is $\frac{1}{6}$ of the whole prism."

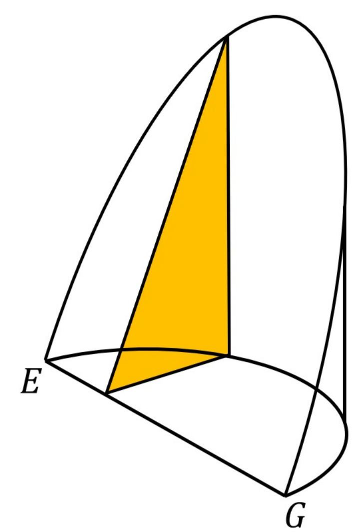
See the figure below. In these notes, we consider the special case were the "whole prism" is a cube with base $ABCD$ (as opposed to a taller or shorter right prism with square base). We take line segment AB to have length 2 (so that the cylinder has radius 1); we could choose any scale, but this special case makes proof of the result using calculus a bit easier. The volume of the cube is then $2^3 = 8$, so that the claim is that the portion of the cylinder cut off by the plane is $(1/6)(8) = 4/3$.



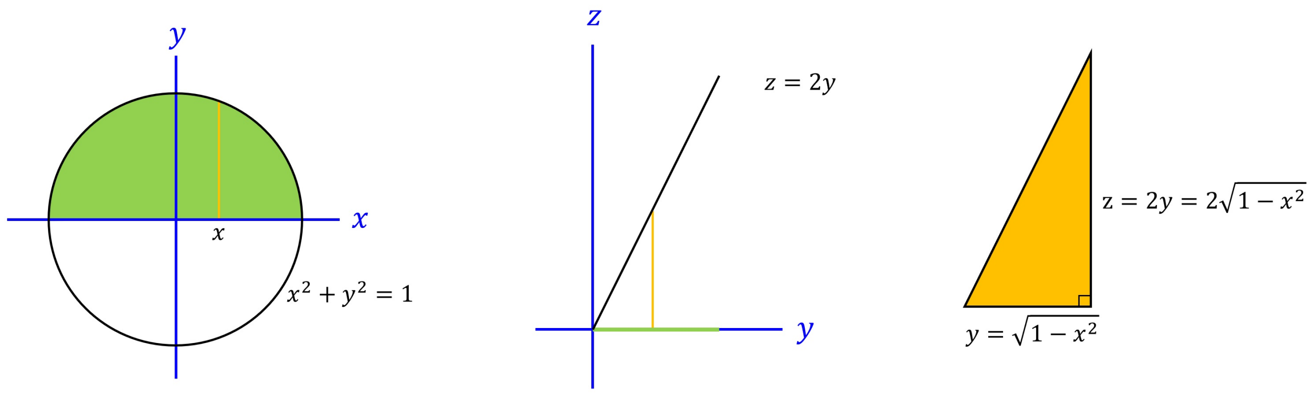
These are Figures 8.1 and 8.2 from *The Archimedes Codex* (with labels of points added). The figure on the left is the part of the solid of interest.

The portion of the cylinder cut off by the plane (the "solid of interest" in the figure above) is described as a "fingernail-like shape" on page 189 of *The Archimedes Codex*, and as "a hoof" in S. Gray, D. Ding, G. Gordillo, S. Landsberger, and C. Waldman's "The Method of Archimedes: Propositions 13 and 14," *Notices of the American Mathematical Society*, **62**(9) (2015). This paper is online on the American Mathematical Society webpage (accessed 11/5/2023) and includes a computation of the volume of the solid using triple integrals (the triple integral implies the integral involving a single variable which we give next). In Calculus 2 (MATH 1920) we define the *volume* of a solid which has a cross-sectional area of $A(x)$ from $x = a$ to $x = b$ to be $V = \int_a^b A(x)\,dx$ (see my online notes for Calculus 2 on Section 6.1. Volumes Using Cross-Sections).





In the figure above, we introduce an xy-coordinate system and illustrate a typical "dx slice" (as we shall call it; granted this is imprecise and informal).



In the images above, we have the xy-plane (left) with the base of the solid in green and the base of the dx slice in orange. The yz-plane (center) gives the profile of the slanted surface of the solid, a cross section of the base in green, and a cross section of the dx slice in orange. On the right, we have the profile of the dx slice. Since the dx slice (itself a cross section of the volume) is a right triangle with base $\sqrt{1-x^2}$ and height $2\sqrt{1-x^2}$, then the cross sectional area is $A(x) = (1/2)\sqrt{1-x^2}(2\sqrt{1-x^2}) = 1 - x^2$. So by the definition of volume from Calculus 2, we have that the solid has volume

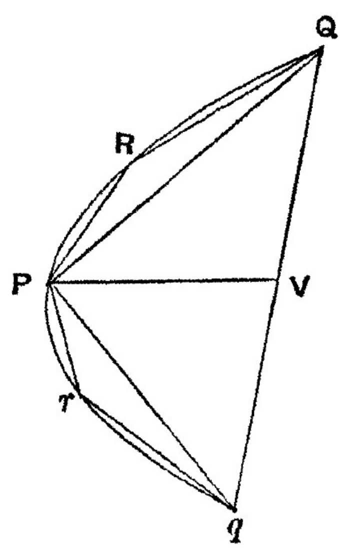
$$V = \int_a^b A(x)\,dx = \int_{-1}^{1} 1 - x^2\,dx = \left(x - \frac{1}{3}x^3\right)\Big|_{-1}^{1} = \frac{4}{3}.$$

As mentioned above, this value confirms Archimedes Proposition 14. As in Note AM2.C, we again have a calculus-based justification of a claim of Archimedes. We now turn to the way Archimedes approached the problem. This note is based on pages 188–192 of *The Archimedes Codex*.

Note AM2.G. In *The Archimedes Codex*, the state of the understanding of the history of Greek mathematics, as of January 2001, is described as (see page 184):

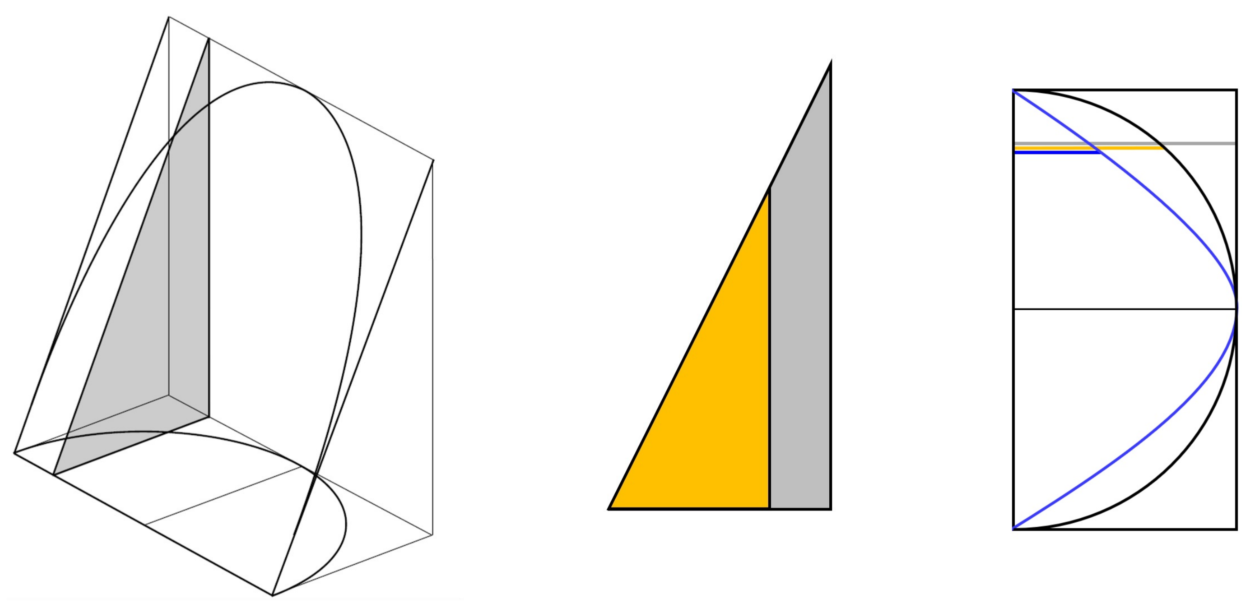
> "The Greeks invented mathematics as a precise, rigorous science. They avoided paradox and mistakes. In doing so, they also avoided the pitfall of infinity. Their science was based on numbers that can be as big as you wish, or as small as you wish, but never *infinitely* big or small. Numbers that are as big or small as you wish are known as 'potentially infinite,' instead of actually infinite. The Greeks did not use actual infinity."

This topic of potential versus actual infinity will be addressed here in the setting of integration and we will see that, in fact, Archimedes manipulated actual infinity. This discovery came from analysis of the palimpsest. By the way, in modern mathematics the study of applications of infinity is accomplished in the area of *analysis* and historically this study was put on a rigorous foundation in the 19th century. Big names is establishing this foundation are Augustin Louis Cauchy (August 21, 1789–May 23 1857; he is the one that brings us the ε/δ arguments of calculus), Bernhard Riemann (September 17, 1826–July 20, 1866; you know him from Riemann sums and the Riemann integral in calculus), and Karl Weierstrass (October 21, 1815–February 19, 1897). For more on these three, see Section 13.5. Cauchy and Section 14.10. Weierstrass and Riemann. The idea behind "potential infinity" in the setting of finding the area of a region r involves a claim that the area is an actual value A then showing that, for any given positive tolerance t, there a region s contained in r that such that the area B of s is within the tolerance of area A (i.e., $A - B < t$). In *The Archimedes Codex* (see page 185), this is described as a back-and-forth conversation in which Archimedes claims that A is the area of region r because it is close to area B_1 (the area of a region s_1, a subset of r). But the other party objects that there is still a difference "greater than a grain of sand" between A and B_1. Archimedes then replies that he can find a new region s_2, a subset of r, with an area B_2 closer to A than a grain of sand. The other party then object that there is still a difference "greater than a hair's width" between A and B_2. Archimedes then creates another region s_3, a subset of r, with an area B_3 closer to A than a hair's width, and so on. The hypothetical dialogue goes on indefinitely, and this is potential infinity. The regions constructed by Archimedes can be made arbitrarily close to r, but "never" precisely equal to r. This process is called the *method of exhaustion*, and it is usually credited to Eudoxus of Cnidus (408 BCE–355 BCE); see Section 11.3. Eudoxus' Method of Exhaustion. It is used in Euclid's *Elements*, Book XII in several results concerning volumes (including the fact that the volume of a cone is $V = \pi r^2 h/3$); see Note 5.4.P of Section 5.4. Content of the "Elements". You likely notice the similarity between this and an ε argument from analysis.



Archimedes employs the method of exhaustion in his *Quadrature of the Parabola* when showing the area bounded by a parabolic segment is 4/3 the area of the triangle it determines (as described in Note AM2.C above, and given in the *Method* as Proposition 1 where it is proved using the Law of Balance). This appears as Proposition 24 of *Quadrature of the Parabola* and he packs the area with triangles. The figure above is from Heath's 1897 *The Works of Archimedes* (see page 251) for Proposition 24 and shows some of the triangles inscribed in the parabolic segment. Additional details on this result are given in G. Swain and T. Dence, "Archimedes' Quadrature of the Parabola Revisited," *Mathematics Magazine*, **71**(2), 123–130 (1998). It is available online on the JSTOR website (accessed 11/10/2023). This note is based largely on pages 184 and 185 of *The Archimedes Codex*.

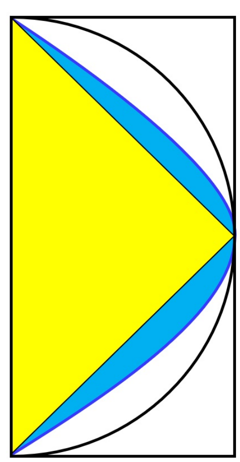
Note AM2.H. We now consider Archimedes' solution to Proposition 14 as revealed by the palimpsest. In the first 13 propositions of the *Method*, Archimedes considers parallel slices of an area or volume which are "taken together" to produce the final area or volume, as discussed above in Note AM2.E. In his proof of Proposition 14, it appears that he is starting another proof using the same approach.



The slice he takes is of the form given in the figure above (based on Figures 8.5, 8.6, and 8.7 of *The Archimedes Codex*). The orange triangle represents the part of the slice that is contained in the solid of interest (as given by the orange triangle of the second figure in Note AM2.F on the right). The gray triangle (part of which is obscured by the orange triangle; middle) is the gray triangle on the left in the figure above. The base of the whole solid on the left is given on the right. The base is a rectangle, the cylinder intersect the base in a semicircle, and the gray triangular slice intersects the base in the gray line segment. Archimedes then uses the two points on the left corners of the base to introduce a parabola on the base (finding the third point using the midpoint of the line joining the other two points, as is done in the figure in Note AM2.B when finding point B there; in our special case, this is easy since the base is a rectangle, and not just a parallelogram), and introduces a line segment parallel to the axis of the parabola, as shown in blue (notice that this is *not* the base of the orange triangle). Archimedes bases his argument on the orange triangle (called "the triangle of the cylinder"), the gray triangle (called "the triangle of the prism"), the gray line segment (called the line of the rectangle"), and the blue line segment (called "the line of the parabola"). In Proposition 13 of the *Method*, Archimedes proves that in this configuration:

> "The area of the triangle of the prism is to the triangle of the cylinder as the line of the rectangle is to the line of the parabola." [As stated on page 195 of *The Archimedes Codex*.]
>
> "($\triangle$ in prism) : ($\triangle$ in portion of cylinder) $= MN : ML =$ (straight line in rect. DG) : (straight line in parabola)." [As stated in Heath's 1912 *The Method of Archimedes* on page 42.]

At this stage of the proof of Proposition 14, unfortunately, the writing in the palimpsest is not legible. This results in a gap in the argument. Johan Heiberg, when reading the palimpsest in 1906, is able to read the proof of Proposition 14 up to this point. After the gap, the next part that he is able to read is (see *The Archimedes Codex*, page 195): "The volume of the triangular prism is to the volume of the cylindrical cut as the area of the entire rectangle is to the area of the entire parabolic segment." Presumably, Archimedes has arrived at this by "taking together" the slices, as he is known to have done in his previous propositions in the *Method* (see Note AM2.D). But maybe not, as we discuss further below... Now the area bounded by the parabola in light blue, part of which is obscured by the yellow triangle (see the figure below), is 4/3 of the area of the yellow triangle inscribed in the parabola by Proposition 1 of the *Method*. The area of the rectangular base is twice that area of the yellow triangle, so the "area of the entire rectangle" to the "area of the entire parabolic segment" is $(4/3) : 2$ or $(4/3)/2 = 2/3$. That is, the area bounded by the parabola is 2/3 the area of the rectangular base, or the area of the rectangular base is 3/2 of the area bounded by the parabola. Hence, by Archimedes' claim, the volume of the triangular prism is 3/2 of the volume of the cylindrical cut (i.e., the solid of interest).



Now the triangular prism containing the solid of interest (a slice of the triangular prism is taken in the figure in Note AM2.H on the left) is 1/4 of the cube containing the solid of interest (see the first figure, left, in Note AM2.F). So the volume of the entire cube 4 times the volume of the triangular prism, and the triangular prism is 3/2 the volume of the solid of interest. We now have that the cube is $4 \times 3/2 = 6$ times the volume of the solid of interest. That is, the volume of the "cylinder cut off by the plane" (i.e., the solid of interest) is 1/6 the volume of the cube (or, more generally, the "whole prism," by which Archimedes in Proposition 14 means a right prism with a square base, but not necessarily a cube). This completes Archimedes' proof. But what about that missing part of his proof? This note is based on pages 192–196 of *The Archimedes Codex*.

Note AM2.I. Heiberg's failure to record Archimedes' argument for the crucial step in his proof of Proposition 14 is easily explained by viewing the palimpsest as it is today. The page containing the argument is largely illegible. Even with ultraviolet light, it cannot be read. How was it that Archimedes transition from infinitely many proportions to single proportion representing a "sum" of the infinitely many proportions (more appropriately, an "integral" of the infinitely many proportions)? The team studying the palimpsest was persistent. Reviel Netz (one of the coauthors of *The Archimedes Palimpsest*) and Ken Saito (a well-known historian of mathematics) made their first breakthrough when Noel identified the letters $\varepsilon\gamma\epsilon\theta$, corresponding to the English "egeth." The two decided that this must be part of the Greek word *megethos*, meaning "magnitude." This would not be expected in a specific "concrete" calculation. The Greek interpretation of the term in Archimedes' time is given on page 197 of *The Archimedes Codex* as:

> "The word 'magnitude,' with it generality, is appropriate not in a concrete, geometrical context, but in a more abstract context such as in the study on the theory of proportions or of magnitudes. It was as if, in the middle of a calculation with concrete numbers, the text moved to a discussion of the principles of calculations as such."

Speculation started that Archimedes had gone beyond potential infinity to actual infinity. This gave great motivation for the imaging team in the palimpsest project to start with this part of the palimpsest. A high-resolution, sharp digital image, made with UV light, of the single piece of the palimpsest involving the bifolio 105-110 of the prayer book was made. With this image, Netz was able to pick out "megethos' ("magnitude") in several places on the page, as well as words referring to various geometric objects. This revealed that the digital images would be extremely useful in reading parts of the *Method* that were otherwise illegible. The most revealing term that Netz found was *isos plethei*, or "equal in magnitude." The page was "peppered with 'equal in magnitude' " (as Netz puts it in *The Archimedes Codex* on page 201). The Greeks used the term "equal in magnitude" to indicate that the number of objects in two sets are the same. Today, we would say the two sets have the same *cardinality*. Archimedes' argument is that the set of cross sections of the cube is equal in cardinality to the set of lines in the rectangle. The cross sections of the cube are given by the gray triangles above (actually, these are 1/4 of cross sections of the cube), and the lines in the rectangle are the grey lines in the figure above (see Note AM2.H). He is using this equality in cardinality to justify passing a proportion between areas of triangles and lengths of lines (as given in Method 13) to a proportion between volumes of solids and areas of regions. The "volumes of solids" here are the volumes of the triangular prism (which is 1/4 the volume of the cube) and the volume of the cylindrical cut (i.e., the solid of interest), and the "areas of regions" here are the areas of the rectangle and the area of the parabolic segment (these ratios are shown to be 3/2 in Note AM2.H; this is then multiplied by 4 to get the ratio of 6 : 1, or 1 : 6 as given in Proposition 14). Netz summarizes Archimedes' argument as (see pages 201 and 202 of *The Archimedes Codex*):

> "And Archimedes pointed out that the number of triangles of which the prism was made was the same as the number of lines of which the rectangle was made. Surely he meant this to be verified by the fact that there was a **onto-to-one relationship**. ... Archimedes repeated this type of statement three times: he went through the various configurations produced by the slices, showing which set was equal in multitude to which set. ... of course, those equalities of number were like nothing else we every knew from Greek mathematics. ... **Archimedes was explicitly calculating with infinitely great numbers.**" [Emphasis added]

This note is based on pages 196–201 of *The Archimedes Codex*.

Note AM2.J. In dealing with the equality of the two infinite sets in cardinality (or "equality in magnitude"), Archimedes is using the one-to-one correspondence between the triangular cross sections of the volumes and the line segment cross sections of the areas. The one-to-one correspondence is given by the fact that each triangle sits on one of the lines; each gray triangle sits on a gray line, and each orange triangle sets on a blue line in the first figure of Note AM2.H. In Mathematical Reasoning (MATH 3000), two sets are of equal cardinality (or are equal in magnitude) or are *equipotent* if there is a bijection between them (i.e., if the two sets are in a one-to-one correspondence); see Definition 4.1 in Section 4.1. Cardinality; Fundamental Counting Principles of my online Mathematical Reasoning notes. This is basically what Archimedes did, but this definition of equipotence dates from the late 1800s. Georg Cantor (March 3, 1845–January 6, 1918) is a key figure in the development of cardinalities of sets and cardinal numbers. A brief biography of Cantor is given in my Mathematical Reasoning notes on Section 4.2. Comparing Sets, Finite or Infinite. Notice that Archimedes might have simply assumed that all infinite sets can be put in a one-to-one correspondence with each other, but he did not do this. Instead, he focused on approaching the slices in such a way that the one-to-one correspondence was given by their relative positions. In fact, some infinite sets are "larger" than others. For example, both sets $\mathbb{N}$ and $\mathbb{R}$ are infinite, but $\mathbb{R}$ is a set that is strictly larger than $\mathbb{N}$ (that is, $\mathbb{R}$ is of a cardinality that is a larger infinity than the infinite cardinality of $\mathbb{N}$). See my online notes on Mathematical Reasoning on Section 4.3. Countable and Uncountable Sets. In *The Archimedes Codex* (pages 202 and 203), the lessons learned by deciphering bifolio 105-110 include:

1. "[Archimedes relied] on certain principles of summation. This means that he was already making a step toward modern calculus and was not merely anticipating it in some naive way."

2. ". . . Archimedes calculated with actual infinities in direct opposition to everything historians of mathematics have always believed about their discipline. Actual infinities were known already to the ancient Greeks."

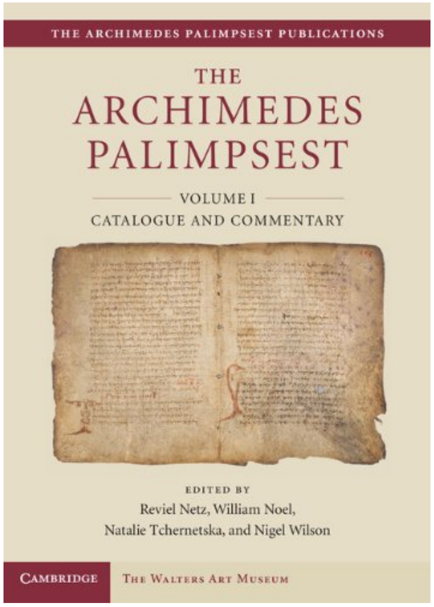
3. "... we see that this concept of infinity—as with so many others—the genius of Archimedes pointed the way toward the achievements of modern science itself. Back in the third century BC, at Syracuse, Archimedes foresaw a glimpse of Set Theory, the product of the mature mathematics of the late nineteenth century."

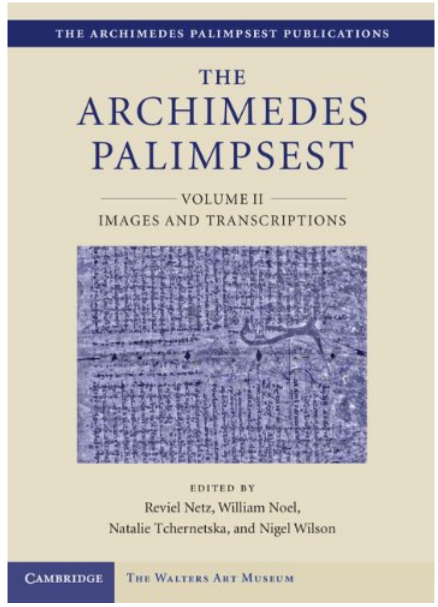
Note AM2.K. We now return to the story of the deciphering of the Archimedes palimpsest. Two imaging teams, one at Johns Hopkins University and the other at the Rochester Institute of Technology, used the technique of "multi-spectral imaging" to extract the Archimedes texts from the palimpsest. For example, when the parchment is illuminated with ultraviolet light it absorbs the light and re-emits it in the blue part of the spectrum. The ink on the parchment obscures the absorption/emission process and, as a consequence, the writing appears dark with the parchment appearing in blue light back lighting the dark writing. But this was just the starting point since it was still hard to distinguish between the writing in the prayer book and the underlying Archimedes work. Using tungsten light, the teams created images in red, green, and blue "channels." In the red channel, the Archimedes text almost disappeared. In this way, the different channels could be combined to produce images with bright parchment, dark prayers, and dark Archimedes print in red. This color difference allowed the Archimedes text to be distinguished from the other writing based on its color. By September of 2001, the teams had a solid approach to generate useful images that could be used to see the work of Archimedes. This note is based on pages 205, 207, 219, and 220 of *The Archimedes Codex*.

Note AM2.L. The work was slow going, though. By autumn of 2003, the palimpsest was still not completely disbounded. The images were good, but not good enough for "Mr. B." The biggest obstacle was provided by the four folios containing the forged paintings (see Note AM.N and Note AM2.A). The multi-spectral imaging was useless to "see through" the paintings to the underlying text. The imaging was eventually approached with the technique of X-ray fluorescence. With this, the object is bathed with X-ray radiation which is absorbed and re-emitted at wavelengths which indicate the composition of the material. The plan was to use this to detect the iron in the ink of the writing, thus showing the writing which is under the forged paintings. Tests of the technique were promising, though it became clear that a very high energy source for the X-rays would be needed. The team turned to the Stanford Linear Accelerator ("SLAC") in California. In March 2006 the team spent two weeks at SLAC producing images using the synchrotron radiation from the accelerator. William Nowell described the experience as (see page 277 or *The Archimedes Codex*):

> "From the moment the scanning started, it was clear that something extraordinary was happening. The charred, stained, and worm-eaten parchment [being scanned] appears on the screen as a dense lattice of Greek characters. I knew that we were seeing, pixel by pixel, line by line, at the Stanford synchrotron, a map of the iron on the page..."

In 2011, the results of the exploration of the Archimedes palimpsest team was published in a two volume set. The title was *The Archimedes Palimpsest* (The Archimedes Palimpsest Publications), Volumes 1 and 2. The editors were Reviel Netz, William Noel,Nigel Wilson, and Natalie Tchernetska and it was published by Cambridge University Press.



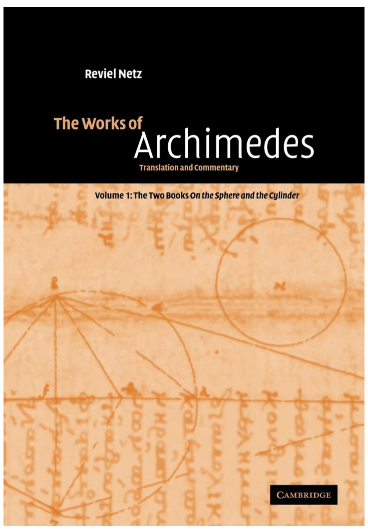


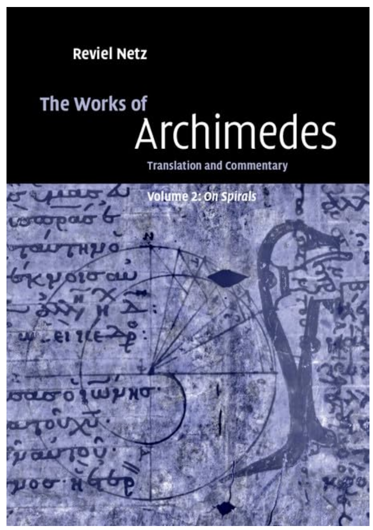
In addition, there is a lengthy Archimedes Palimpsest website (accessed 11/25/2023). This note is based on pages 261, 269, 270, 273, 277 of *The Archimedes Codex*.

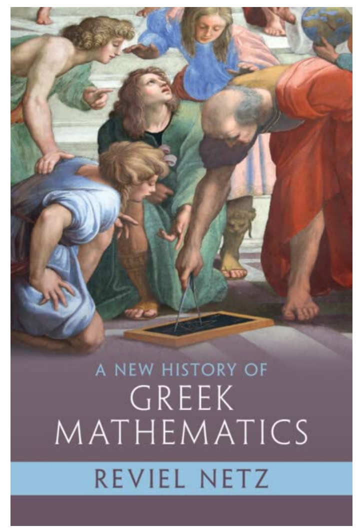
Note. The palimpsest does not consist only of the *Method*. It also includes two previously unknown folios of Archimedes *Floating Bodies*. Based on the information extracted from the palimpsest, new editions of *Stomachion* and *Floating Bodies* are in preparation. Details on these works are given in Section 6.2. Archimedes. In addition, about 30 folios contain writing by someone other than Archimedes. These writings have also lead to new historical insights in nonmathematical areas. In addition, the prayer book itself is of interest. See pages 226–232 of *The Archimedes Codex* for more on this.

Note. Reviel Netz (coauthor of *The Archimedes Codex*) has produced two volumes of Archimedes' work. The first is *The Works of Archimedes: Volume 1, The Two*

Books On the Sphere and the Cylinder: Translation and Commentary (Cambridge University Press, 2009); the "two books" are *On the Sphere and Cylinder I* and *On the Sphere and Cylinder II*. It contains the first English translation of Eutocius's commentary on Archimedes' *On the Sphere and Cylinder*. The second is *The Works of Archimedes: Volume 2, On Spirals: Translation and Commentary* (Cambridge University Press, 2017). It contains the first fully-fledged English translation of *On Spirals*.







Netz also recently published *A New History of Greek Mathematics* (Cambridge University Press, 2022; above right). In the preface he describes this work as an updated version of Thomas Heath's *History of Greek Mathematics* (Oxford, 1921), and not a *replacement* of Heath. Though Heath's work still stands up, it takes more of an encyclopedic approach, whereas Netz gives a historical account and details the conditions and scope of the new science that emerged in the ancient Greek world "providing the tools for modernity" (page xii of his Preface).

Note. Finally, we observe that an episode of the Public Broadcasting System's (PBS) show *NOVA* aired on September 30, 2003 titled *Infinite Secrets*. PBS has a webpage devoted to *Infinite Secrets* which includes links to a transcript of the show, and a "Teacher's Guide" for use in the middle school and high school classroom. There is also a YouTube video of *Infinite Secrets*, though it is of rather low resolution. These two website were accessed 12/3/2023.

Revised: 12/3/2023