Supplement. The Content of Archimedes' Work, Part 1

Note. As mentioned in the notes for Section 6.2. Archimedes, we rely on Chapter XIII of Thomas Heath's *A History of Greek Mathematics, Volume 2* (Oxford: Clarendon Press, 1921), Thomas Heath's *The Works of Archimedes* (Cambridge University Press, 1897) (both of Heath's books are available from Dover Publications), and Reviel Netz and William Noel *The Archimedes Codex: How a Medieval Prayer Book is Revealing the True Genius of Antiquity's Greatest Scientist* (Da Capo Press, 2007) for sources on the work of Archimedes. We also use the recently published book of Reviel Netz, *A New History of Greek Mathematics* (Cambridge University Press, 2022), which was mentioned at the end of Supplement. Archimedes' Method, Part 2.

Note AW.A. In *A History of Greek Mathematics, Volume 2*, Heath gives a chronological listing of Archimedes' works as he viewed it. Netz, based on the letters of introduction which appear at the beginning of many of Archimedes' works, gives the following chronology in his *A New History of Greek Mathematics*:

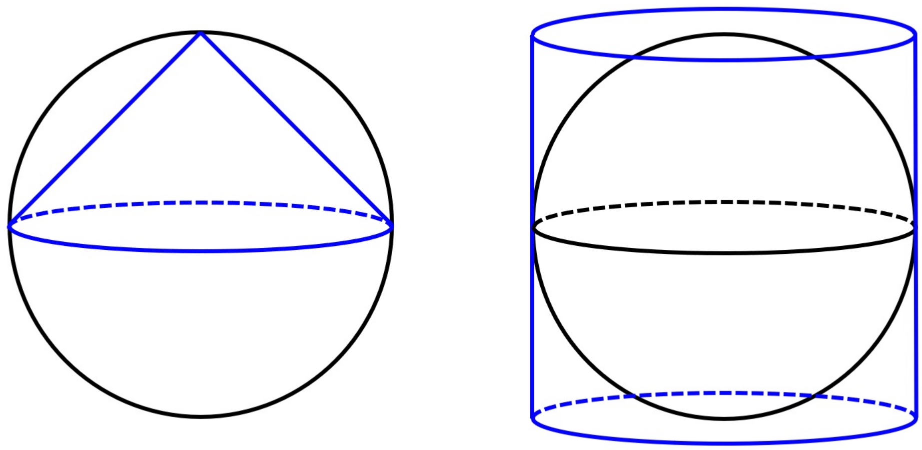
1. *Quadrature of the Parabola*
2. *Sphere and Cylinder* Book I
3. *Sphere and Cylinder* Book II
4. *Spiral Lines*
5. *Conoids and Spheroids*
6. *The Method*

7. *On Floating Bodies*

8. *On Balancing Planes*

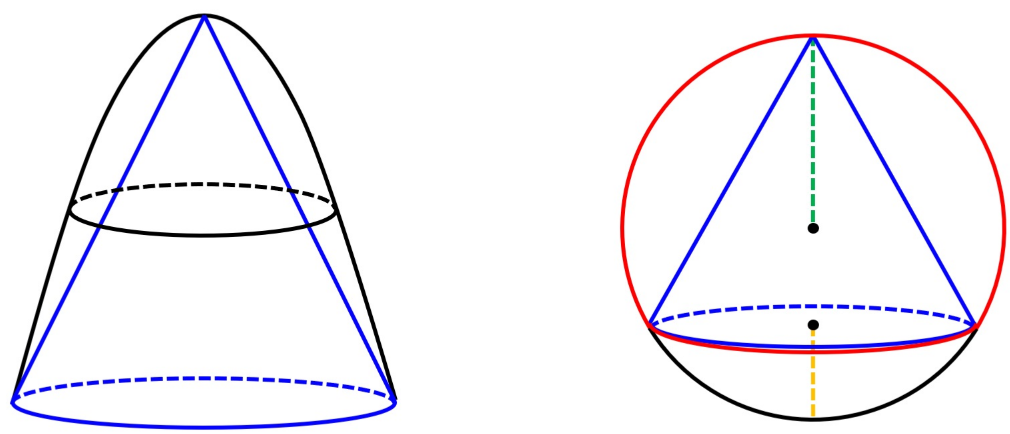
Netz also lists several other works, including *Measurement of Circle*, *Sand-Reckoner*, *Cattle Problem*, and *Stomachion*. In this supplement, we explore the mathematical content of these works. We start with the most interesting work, *The Method*, and then proceed in chronological order through the other works.

Note AW.B. The full title of "*The Method*" is *On Mechanical Theorems, Methods* (communicated) *to Eratosthenes* (Heath' *History, Vol. 2*, pages 27 and 28). The history of how we came to know of the content of this work is explained in Supplement. Archimedes' Method, Part 1. The Proposition 1 of *The Method* involves the area of a region bounded by a parabola and a line; Archimedes refers to this as an "area of a segment of a section of a right-angled cone." His proof technique is explained in detail in both Supplement. Archimedes' Method, Part 2 and the PowerPoint presentation Archimedes: 2,000 Years Ahead of His Time (with an online transcript of the presentation in PDF). His technique involves balancing line segments from the section with those from a triangle (not the triangle mentioned in the theorem, but one related to it). He then "sums" or takes all of the line segments "collectively" to conclude that the area of the region bounded by the parabola and line is equal to 4/3 of the area of the triangle with the same base and height as the region. Proposition 2 concerns the volume of a sphere as it relates to an inscribed cone and circumscribed cylinder, as illustrated below. The volume of the sphere is four times the volume of the cone, and the volume of the cylinder is $1\frac{1}{2}$ times the volume of the sphere. Since the volume of the cylinder is $\pi r^2 h = \pi r^2(2r) = 2\pi r^3$, then it follows that the volume V of the sphere satisfies $1\frac{1}{2}V = 2\pi r^3$ or $V = \frac{4}{3}\pi r^3$, as we expect. With the chronology of Archimedes' work as laid out by Netz, we see that Archimedes already had established this result in his earlier *Sphere and Cylinder*. Proposition 3 is a similar result, but for spheroids (that is, revolutions of an ellipse about its major axis or its minors axis) instead of spheres.



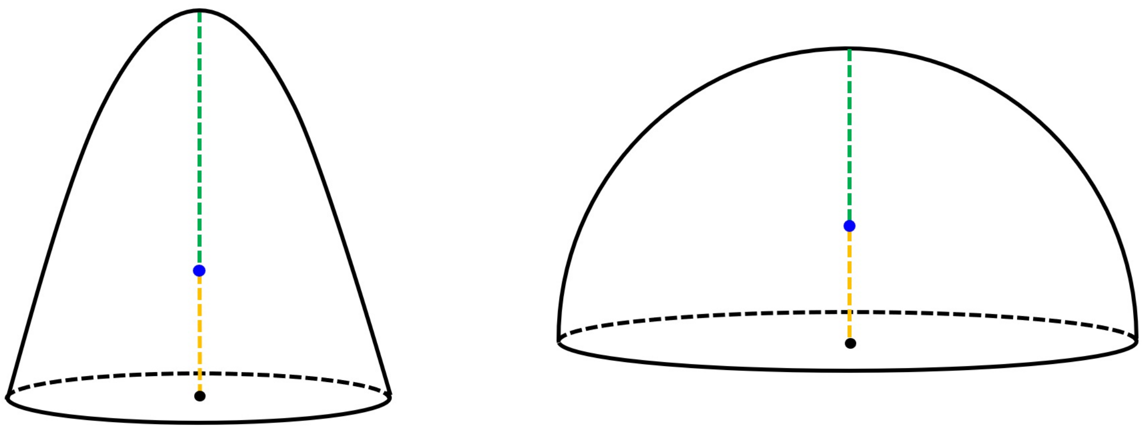
The Method, Proposition 2

Archimedes' proofs of Propositions 2 and 3 are similar to that of his proof of Proposition 1 in that he argues in terms of balancing cross sections. However, in Proposition 2 and 3 the cross sections are two dimensional regions, instead of line segments as in Proposition 1. As described by Heath in his *History, Volume 2* (see page 28), Propositions 4, 7, 8, and 11 concern the volume of a segment cut off, by a plane at right angles to the axis, from a "right-angled conoid" (i.e., paraboloid of revolution), sphere, spheroid (i.e., revolution of an ellipse about an axis), and obtuse-angled conoid (i.e., hyperboloid of revolution) in terms of the cone with the same base and height as the segment. For example Proposition 4, illustrated below, states that the volume of the paraboloid is 4/3 the volume of the inscribed cone. Proposition 7 states that the ratio of the volume of the cut sphere (in red) to the volume of the cone (in blue) equals the ratio of the radius of the sphere plus the length of the complement segment (in green and orange, respectively) to the length of the complement segment (in orange). Each of Propositions 4, 7, 8, and 11 involves the proof technique of Propositions 1 through 3, namely the "mechanical method" of balancing cross sections.



The Method, Proposition 4 (left) and Proposition 7 (right)

As described by Heath in his *History, Volume 2* (see page 28), Propositions 5, 6, 9, and 10 concern the center of gravity (or "centroid") of segments of a paraboloid of revolution, a sphere, and spheroid (i.e., revolution of an ellipse about an axis). For example Proposition 5, illustrated below, states that the center of gravity of the paraboloid cut off by a plane at right angles to the axis lies on the axis at a point (in blue) with the distance from the vertex to the point (given by the segment in green) is twice the distance from the point to the base of the segment of the paraboloid (given by the segment in orange). Proposition 6 states that the center of gravity of a hemisphere lies (in black in the figure below) on the axis of the hemisphere at a point (in blue) with a distance from the top-most point of the hemisphere (given by the segment in green) in a ratio of the distance from the point to the base of the hemisphere (given by the segment in orange) of 5 to 3.



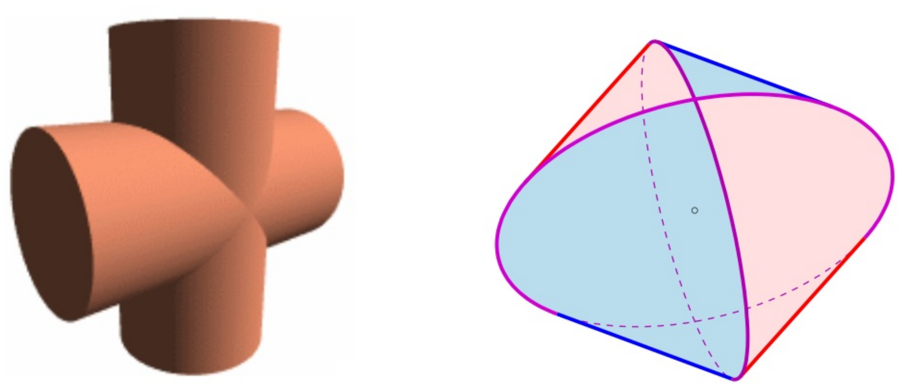
The Method, Proposition 5 (left) and Proposition 6 (right)

Propositions 12 to 14 consider the "hoof" (as Heath calls it) that results by cutting a cylinder by a plane perpendicular to the cylinder (determining a base for the solid) and a second plane passing though a diameter of the base, as illustrated in Supplement. Archimedes' Method, Part 2 (see Note AM2.F). Recall that the volume of the hoof is 1/6 the volume of the cube containing it; the cube is referred to as the "original prism" in Heath's translation of *The Method*. The numbering of the propositions given in Heath's 1912 version of *The Method* seems inconsistent with the number Heath uses in his *History, Volume 2*. Tellingly, Netz in *New History* mentions these results by content, but not by number. For this reason, we refer to "Propositions A, B, C, D" which correspond to the Propositions 12, 13, 14, 15 (and possibly 16) as given in Heath's widely available 1912 version of *the Method* (there is no Proposition 16 in Heath's *The Method*, but he refers so a Proposition 16 in his *History, Volume 2* on pages 28 and 29). Proposition A (12?) gives a mechanical argument based on balancing cross sections, as is done in the previous 11 propositions in *The Method*. Proposition B (13?) gives a second argument described by Netz in *New History* (see page 190) as "less than a rigorous proof and yet involving no mechanics [i.e., no mention of balancing]." Proposition C (14?) is the "famous" proof explored in Supplement. Archimedes' Method, Part 2 (though Heath's 1912 version of *The Method* describes a method of exhaustion proof, further confusing the numbering schemes; the fact that Heiberg and Heath only had access to a fragmentary version of *The Method* may also explain the discrepancies). The Proposition 14 proof is explored in detail in two research papers from the early days of the palimpsest project:

1. R. Netz, K. Saito, and T. Tchernetska, "A New Reading of *Method* Proposition 14: Preliminary Evidence from the Archimedes Palimpsest (Part 1)," *SCIAMVS*, **2** (2002), 9–29.

2. R. Netz, K. Saito, and T. Tchernetska, "A New Reading of *Method* Proposition 14: Preliminary Evidence from the Archimedes Palimpsest (Part 2)," *SCIAMVS*, **3** (2003), 109–125.

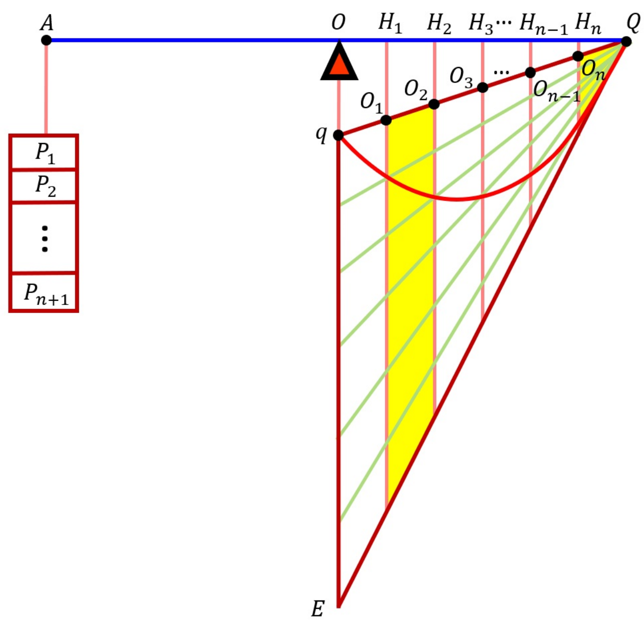
Both of these papers are online on the *SCIAMVS*, Sources and Commentaries in the Exact Sciences, Back Issues webpage as: Part 1 and Part 2.



Intersecting cylinders and the volume common to both; from the Wikipedia page on the Steinmetz Solid (accessed 4/6/2024)

In Proposition D (15 and maybe 16?), Archimedes considers the intersection of two circular cylinders of the same diameter where the axes of the cylinders intersect at a right angle (see the figures above). Archimedes argues (though much of the proof is missing; perhaps this is the cause of the inconsistency in a reference to Proposition 16 also) that the volume common to the two cylinders is 2/3 that of the cube containing the intersection. Archimedes has used two basic techniques in *The Method*. The main technique of proof (and the one on which the title of the work is based) is the use of a mechanical balance applied cross sections to draw conclusions about the compilation of the cross sections. The other technique (used only in the last two or three propositions) is to establish a ratio between cross sections of different objects and then to conclude that the ratio also holds between the objects themselves. Notice that all of these problems can be solved in freshman or sophomore calculus. The area bounded by a parabola and a line is covered in Calculus 1 (MATH 1910) in Section 5.6. Substitution and Area Between Curves. Centers of mass (or centroids) is covered in Calculus 2 (MATH 1920) in Section 6.6. Moments and Centers of Mass, and volumes of solids of revolution are covered in Section 6.1. Volumes Using Cross-Sections and Section 6.2. Volumes Using Cylindrical Shells. In Calculus 3 (MATH 2110) volumes can be found with double integrals in Section 15.2. Double Integrals over General Regions and found with triple integrals in Section 15.5. Triple Integrals in Rectangular Form (in fact, finding the volume contained in the intersection of two cylinders is a standard triple integral problem). Centers of mass for three-dimensional objects are covered in Section 15.6. Moments and Centers of Mass.

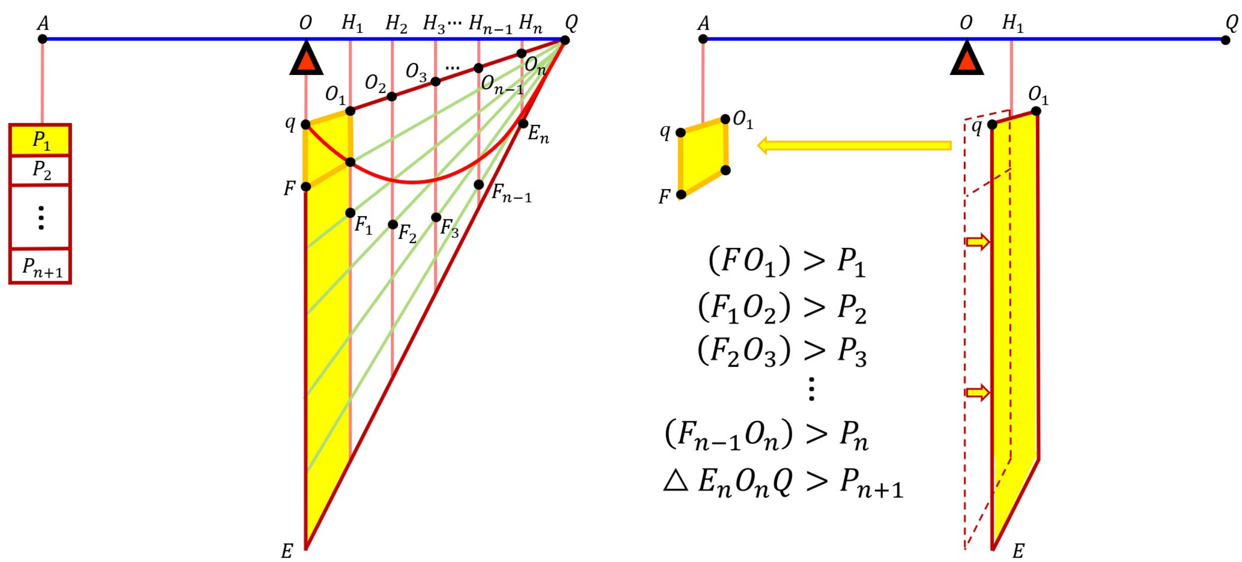
Note AW.C. *Quadrature of the Parabola* consists of 24 propositions. As the title suggests, the work considers the area bounded by a parabola and a straight line (i.e., a "parabolic segment"). This is the same problem considered in *The Method*. In that source, Archimedes takes cross sections of the area (which are line segments) and balances them with a lever by line segments of a convenient length. Then taking the line segments collectively, he balances the parabolic segment with the triangle *related* to the it. The location of the the fulcrum in this balancing sets up a ratio between the areas, from which he deduces that the parabolic segment has an area of 4/3 of the area of a triangle with the same base and height as the parabolic segment. This is spelled out in detail in Supplement. Archimedes' Method, Part 2 (see Note AM2.D) and Archimedes: 2,000 Years Ahead of His Time (in PowerPoint; see also the transcript for this presentation). In *Quadrature of the Parabola*, Archimedes gives two arguments for this same result. In Propositions 1–17, he gives another mechanical argument, but this time using little trapezoidal slices of the region followed by (effectively) the taking of a limit! In this way, he is approaching the problem in the same way we would in modern Calculus 1 (MATH 1910); for such a problem worked in a very similar way, see my online Calculus 1 notes on Section 5.2. Sigma Notation and Limits of Finite Sums and notice Example 5.2.5 (which is work in detail in the Beamer supplement for this material). Archimedes then follows up with a method of exhaustion proof. In Propositions 18–24 he gives a purely geometric solution (without appeal to mechanics and balancing), which again he confirms by the method of exhaustion. In the first proof, Archimedes considers the configuration given in the figure below. The parabola (in light red) is bounded by the line segment qQ (in dark red). He introduces the "lever" AOQ, where O is at the midpoint of AOQ (where a fulcrum is placed). He starts by dividing qQ into equal parts at points $O_1, O_2, \ldots, O_n$, as shown. In Propositions 6–13, it is proved that if the trapezoid in yellow (for example) is suspended from points H_1 and H_2 and an area P suspended from point A balances the trapezoid, then it will take a greater area than P to balance the same trapezoid when suspended from H_2, and it will take a lesser area than P to balance the same trapezoid when suspended from H_1. The same type of result holds for the totality of the n trapezoids, as given, bounded between lines qQ and EQ. Label the n areas which balance the trapezoids as $P_1, P_2, \ldots, P_n$. In Propositions 14 and 15, it is similarly shown that the yellow triangle on the right can similarly be balanced by an area of P_{n+1} suspended at point A. In this way, triangle EqQ is balanced by area $P_1 + P_2 + \cdots + P_{n+1}$.



Based on Heath's *History, Volume 2*, page 87.

Now the center of gravity of a triangle lies on the intersection of the line segments which join an angle to the middle point of the opposite side, as is shown in Propositions 13 and 14 of Archimedes' *On Balancing Planes* (also known as *On* *the Equilibrium of Planes*). The point of intersection lies 1/3 of the way along a line segment as measured from the side it intersects. So in the diagram above, the center of gravity of triangle EqQ lies 1/3 of the way from point O to point Q (since the 1/3 distance along the bisecting line segment intersecting side qE projects onto segment OQ in a proportional way). Since triangle EqQ balances the area $P_1 + P_2 + \cdots + P_{n+1}$ by placing the triangle 1/3 the distance from the fulcrum as $P_1 + P_2 + \cdots + P_{n+1}$ is placed, then we must have that the area of triangle EqQ is 3 times the area $P_1 + P_2 + \cdots + P_{n+1}$. That is,

$$P_1 + P_2 + \cdots + P_{n+1} = \frac{1}{3}\triangle EqQ.$$



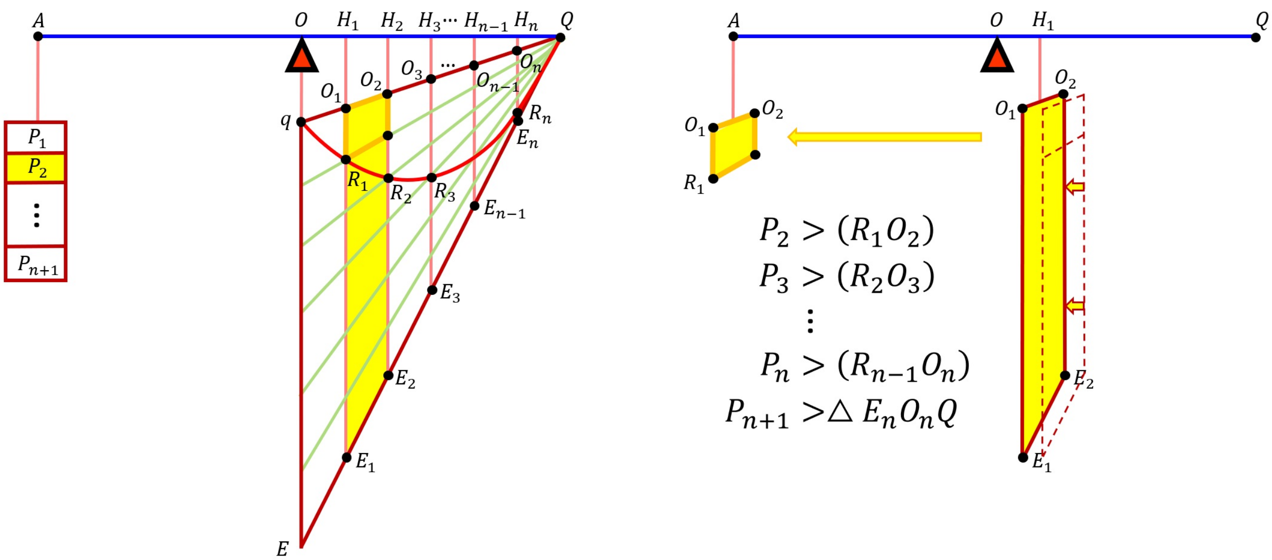
Based on properties of the parabola (given in Proposition 5), Archimedes showed that

$$AO : OH_1 = (\text{area of trapezoid } EO_1) : (\text{area of trapezoid } FO_1).$$

This means that trapezoid FO_1 suspended at A balances trapezoid EO_1 suspended at H_1. Now P_1 balances EO_1 where EO_1 is originally located (with its right-hand side suspended at H_1). So we have the area of trapezoid FO_1 is greater than area P_1: $(FO_1) > P_1$. Similarly, $(F_1O_2) > P_2$, and so forth. In a similar way,

$$AO : OH_1 = (\text{area of trapezoid } E_1O_2) : (\text{area of trapezoid } R_1O_2)$$

from which we find $P_2 > (R_1O_2)$. These two patterns also hold for the right-most part of the segment and we have $\triangle E_nO_nQ > P_{n+1} > \triangle R_nO_nQ$.



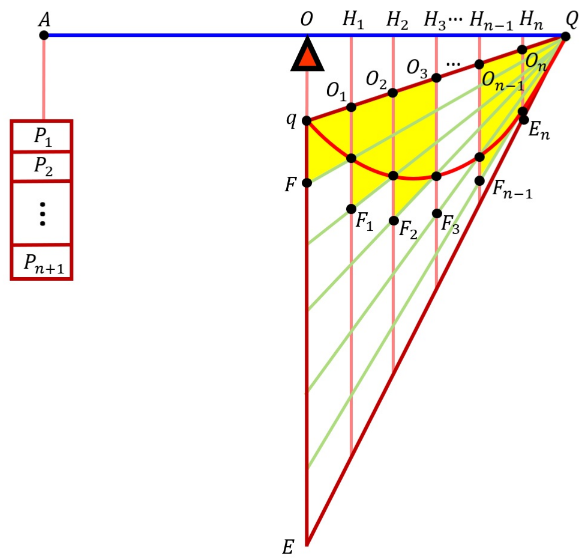
Summing these gives

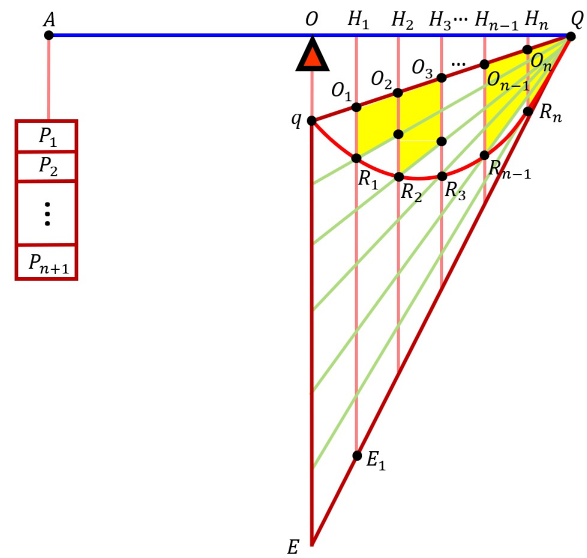
$$(R_1O_2) + (R_2O_3) + \cdots + (R_{n-1}O_n) + \triangle R_nO_nQ < P_1 + P_2 + \cdots + P_n + P_{n+1}$$

$$< (FO_1) + (F_1O_2) + \cdots + (F_{n-1}O_n) + \triangle E_nO_nq$$

(we didn't get a lower bound on P_1, but the inequality still holds since $0 < P_1$). That is, we now have a collection of $n-1$ trapezoids and one triangle which are *inscribed* in the parabolic segment, and n trapezoids and a triangle that are *circumscribed* in the parabolic segment (see the figure below). Intuitively, by increasing the number of such objects (i.e., by increasing n), since their widths are the same (namely, OQ/n, because their widths are determined from the uniformly distributed points

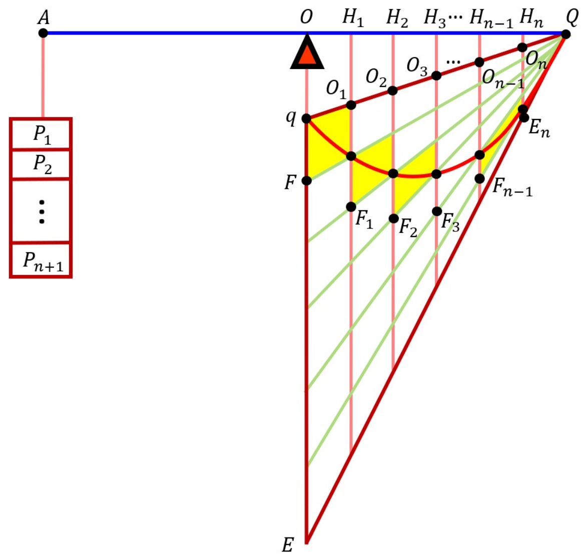
$q, O_1, O_2, \ldots, O_n, Q$ along line segment qQ), both the inscribed and circumscribed areas should both get close to the area of the parabolic segment (one close from below, the other close from above).

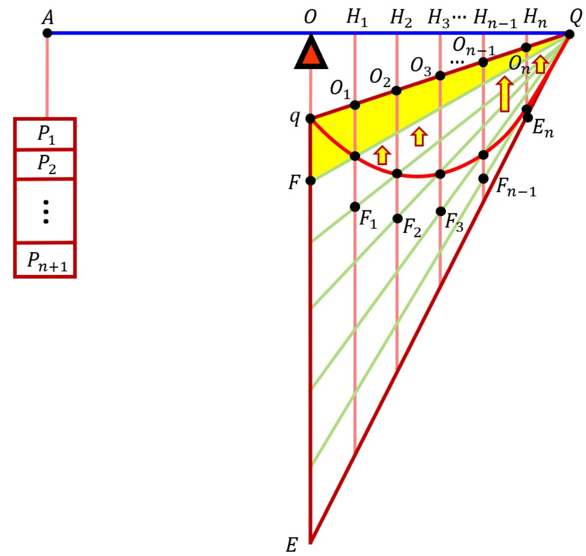




"In order to enable the method [of exhaustion] to be applied, it has only to be proved that, by increasing the number of parts in Qq sufficiently, the difference between the circumscribed and inscribed figures can be made as small as we please" (Heath's *History, Volume 2, page 88*). Based on Proposition 5 again, Archimedes shows that the line segments between point Q and line segment qE cut the line segments $O_1R_1, O_2R_2, \ldots, O_nR_n$ into equal length subsegments (O_1R_1 is not cut, O_2R_2 is bisected, O_3R_3 is trisected, etc.). So in each "column" of trapezoids, the width of all trapezoids are equal (namely, OQ/n) and the heights are the same (namely, $O + iR_i/i$ for $i = 1, 2, \ldots, n$). So if we take the difference of the circumscribed trapezoids and the inscribed trapezoids then we get an area equal to that of $\triangle FqQ$ (see the figure below). This area can be small as we please by making the number n sufficiently large (since the even spacing of points of intersection on line segment

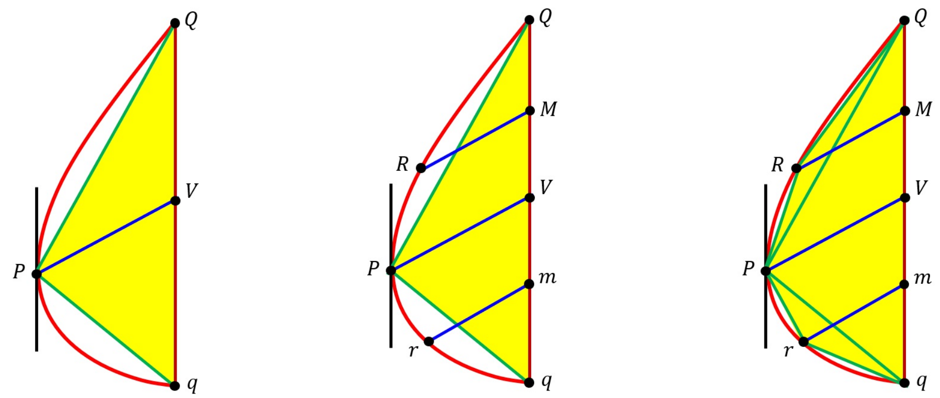
qE implies that increasing n results in decreased distance qF and hence decreased area of $\triangle FqQ$).





That is, the *method of exhaustion* shows that the area of the parabolic segment is $P_1 + P_2 + \cdots + P_{n+1} = \frac{1}{3}\triangle EqQ$. In Proposition 17 (and also Supplement. Archimedes' Method, Part 2; see the last part of Note AM2.D). This implies the area of the parabolic segment is 4/3 the area of a triangle with the same base and height, as claimed in Proposition 16. Notice how closely Archimedes' approach to this problem resembles seen in Calculus 1 (MATH 1910). The sums of the inscribed and circumscribed trapezoids, followed by taking a limit as $n \to \infty$ (which Archimedes did not do, but his "small as we please" comment is equivalent to taking such a limit). The main difference is that in Calculus 1, Riemann sums depend on areas of rectangles instead of trapezoids (see my online Calculus 1 notes on Section 5.2. Sigma Notation and Limits of Finite Sums and notice Figure 5.9). Notice that even though Archimedes has used trapezoids, his approach is not directly related to the numerical technique known as the *Trapezoid Rule*; see my online Calculus

2 (MATH 1920) notes on Section 8.6. Numerical Integration. Archimedes gives his second basic computation of the area of the parabolic segment with a "purely geometric" solution in Propositions 18–24. For parabolic segment given in the figure below cut by line segment Qq, find a tangent to the parabola that is parallel to Qq and let P be the point of tangency (left). The resulting triangle, $\triangle QPq$, has the same base and height as the parabolic segment (this follows from Proposition 1). So we wish to show that the area of the parabolic segment is $\frac{4}{3}(\triangle QPq)$. Let V be the midpoint of Qq and introduce line segment PV (left).



Next, the process is repeated on the parabolic sub-segments determined by sides PQ and Pq in $\triangle PQq$. Let M be the midpoint of QV and let m be the midpoint of Vq. Construct parallels to PV through each of points M and m. Let points R and r be the intersections of these with the parabola, respectively (middle). Then consider triangles $\triangle PQR$ and $\triangle Pqr$, which are inscribed between the parabola and $\triangle PQq$ (right). In Proposition 21, Archimedes shows that

$$(\triangle PQR + \triangle Pqr) = \frac{1}{4}(\triangle PQq).$$

Next, four triangles are inscribed between the parabola and triangles $\triangle PQR$ and $\triangle Pqr$ (on edges QR, RP, Pr, and rq). These four triangles have areas summing to (as above):

$$\frac{1}{4}(\triangle PQR) + \frac{1}{4}(\triangle Pqr) = \frac{1}{4}(\triangle PQR + \triangle Pqr) = \frac{1}{4}\left(\frac{1}{4}(\triangle PQq)\right) = \frac{1}{4^2}(\triangle PQq).$$

Iterating this process gives a sequence of triangles of total area (as shown in Proposition 22)

$$\left(1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots\right)(\triangle PQq).$$

In Proposition 23, Archimedes gives a geometric proof that

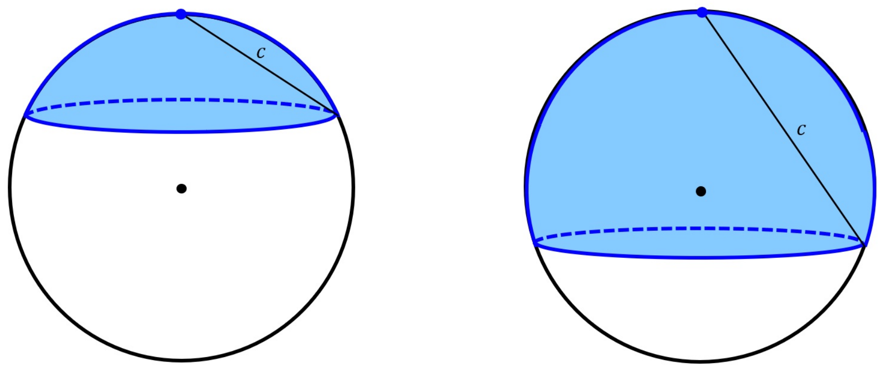
$$1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots + \frac{1}{4^{n-1}} = \frac{1 - 1/4^n}{1 - 1/4}.$$

As opposed to taking a limit (which will not exist, formally, for another 2,000-odd years), Archimedes argues in Proposition 24 that the area of the parabolic segment cannot be less than $\frac{4}{3}(\triangle PQq)$ and cannot be greater than $\frac{4}{3}(\triangle PQq)$. That is, he gives a proof by contradiction (or "*reductio ad absurdum*," in Latin).

Note AW.D. *Sphere and Cylinder* Book I starts with a letter to Dositheus which describes the main results. He describes his results as "certain theorem not hitherto demonstrated...and I have worked out the proofs of them" (Heath's *Works of Archimedes*, page 1). Book I includes 44 propositions. The main results are (largely quoting from Heath's *History, Volume 2*, page 34):

1. The surface area of a sphere is four times that of a great circle of the sphere (or $S = 4\pi^2$). This is Proposition 33.

2. The surface of any segment of a sphere (also called a "spherical section") is equal to a circle the radius of which is equal to the straight line drawn from the vertex of the segment to a point on the circumference of the base of the segment. This is covered in Propositions 42 and 43.



3. The volume of a cylinder circumscribing a sphere and with height equal to the diameter of the sphere is $\frac{3}{2}$ of the volume of the sphere (or the volume of the sphere is $\frac{2}{3}\pi r^2(2r) = \frac{4}{3}\pi r^3$). This is a corollary to Proposition 34.

4. The surface of the circumscribing cylinder including its bases is also $\frac{3}{2}$ of the surface of the sphere. This is also in the corollary to Proposition 34.

In fact, some of these appear in Eves' Problem Studies. Problem Study 6.2(a) requires a computational verification of (3) and (4), using the well-established formulas for the area and volume of a sphere and cylinder. Problem Study 6.2(c) requires a proof of (2), based on an another assumed result about the area of a segment of a sphere. Book I starts with six definitions, including those for a *solid sector*, *solid rhombus* (this is two right circular cones with the same base radius, but possible different heights, glued together at the bases in a way that they do not otherwise intersect [their vertices are on opposite sides of the plane containing their common base) , and "concave" lines (by which he means curves) and surfaces. He then states five assumptions. The first is (we quote from Heath's *History, Volume 2*): "Of all lines which have the same extremities the straight line is the least."

Since the term "line" is used generally to mean an infinite line, a finite line segment, or any continuous curve, what Archimedes is assuming is that the shortest distance between two points is a straight line! The fifth assumption is: "Further, of unequal lines, unequal surfaces, and unequal solids, the greater exceeds the less by such a magnitude as, when added to itself, can be made to exceed any assigned magnitude among those which are comparable with (it and with) one another." Archimedes idea here is that any *quantity* (this is how the "number" concept would be determined at the time) can be repeated enough times to produce as large an amount as desired. This is a property of the real numbers, and you see it in Analysis 1 (MATH 4217/5217) as:

The Archimedean Principle.

If $a, b \in \mathbb{R}$ and $a > 0$, then there is a natural number $n \in \mathbb{N}$ such that $na > b$.

See my online notes for Analysis 1 on Section 1.3. The Completeness Axiom and notice Theorem 1-18. This is also addressed in supplemental notes for Analysis 1 on Supplement. The Real Numbers are the Unique Complete Ordered Field; notice the definition of an *Archimedean ordered field*, Theorem 1.4.3, and Theorem 2.1.A. It is Archimedes' fifth assumption in *Sphere and Cylinder* Book I that is recognized in naming of these results. (Though the result is similar to Euclid's Definition 4 in Book V of the *Elements*: "Magnitudes are said to have a ratio to one another which can, when multiplied [i.e., added to themselves multiples of times], exceed one another." It is in *Sphere and Cylinder* that Archimedes gives ("for the first time") several of the formula, with which we are familiar, concerning areas and volumes of cylinders, cones, and spheres. Two examples (are the following).

Proposition 13. The surface of any right cylinder excluding the base is equal to a circle whose radius is a mean proportional between the side (i.e., a generator] of the cylinder and the diameter of its base.

With the cylinder having radius r and height h (so that the side/generator of the cylinder is a line segment of length h), then the mean proportional of the side and diameter of the base is $\sqrt{(h)(2r)}$ (recall that the mean proportional of a and b is $\sqrt{ab}$). So Proposition 13 claims that the surface area of a right cylinder (excluding base) is equal to the area of a circle of radius $\sqrt{2rh}$, namely $\pi\left(\sqrt{2hr}\right)^2 = 2\pi rh$, as expected.

Proposition 14. The surface of any isosceles cone excluding the base is equal to a circle whose radius is a mean proportional between the side of the cone (i.e., a generator) and the radius of the circle which is the base of the cone.

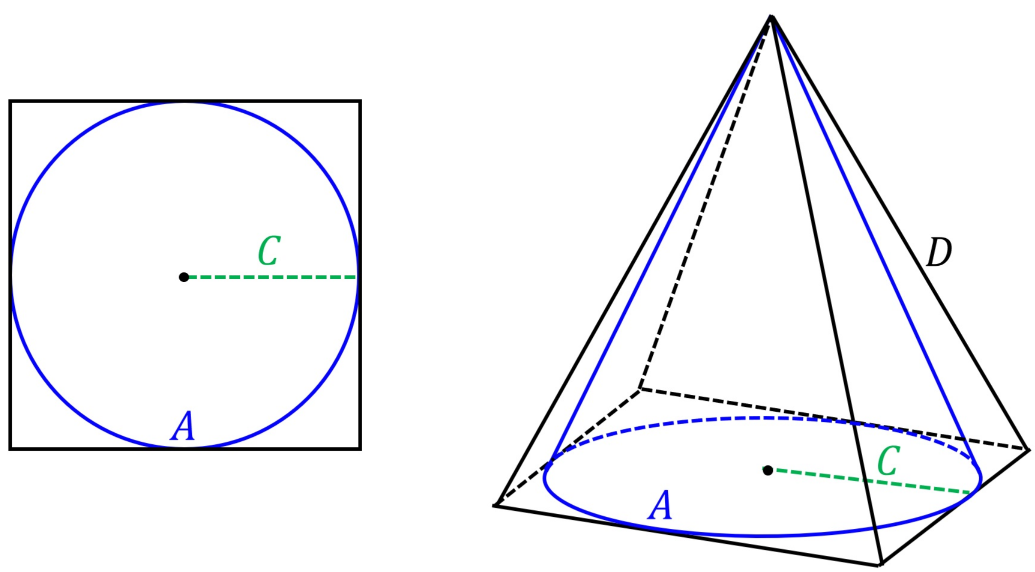
With the cone having base radius r and height h (so that the side/generator of the cone is a line segment of length $\sqrt{r^2+h^2}$), the mean proportional between the side and radius is $\sqrt{(\sqrt{r^2+h^2})(h)}$. So Proposition 14 claims that the surface area of an isosceles cone (excluding base) is equal to the area of a circle of radius, namely $\pi\left(\sqrt{(\sqrt{r^2+h^2})(h)}\right)^2 = \pi h\sqrt{r^2+h^2}$, as expected.

Proof of Proposition 14. Let A be the circular base of the cone, let C be the radius of A, let D be the length of a generator of the cone, let E be the mean proportional of C and D, $E = \sqrt{CD}$, and let B be a circle with radius E. So we need to prove that $S = B$, where S is the surface area of the cone (excluding the base). Archimedes uses the method of exhaustion to show that both $B < S$ and $B > S$ are impossible. Here, we give his argument for the first case. ASSUME $B < S$. By Proposition 5, we can circumscribe a regular polygon around circle B

and inscribe a similar polygon to it inside circle B such that the ratio of the

$$\begin{pmatrix} \text{area of the} \\ \text{circumscribed polygon} \end{pmatrix} : \begin{pmatrix} \text{area of theinscribed polygon} \end{pmatrix} < S : B \qquad (*)$$

(where $S/B > 1$). Circumscribed circle A (the base of the cone) with yet another regular polygon similar to the other (since these are regular polygons, the similarity is reflected in the number of sides). Based on this last regular polygon, circumscribes on the base of the cone, create a pyramid that circumscribes the cone and has the polygon as its base (see the figure below, in the case that the regular polygons are squares).



The area of an n-gon circumscribed about a circle of radius r is $nr^2 \tan(\pi/n)$ and the length of a side is $2r\tan(\pi/n)$ (we accept these as given). So

$$\begin{pmatrix} \text{area of the} \\ \text{polygon about } A \end{pmatrix} : \begin{pmatrix} \text{area of thepolygon about } B \end{pmatrix} = C^2 : E^2 = C^2 : (\sqrt{CD})^2$$

$$= C : D = \begin{pmatrix} \text{area of the} \\ \text{polygon about } A \end{pmatrix} : \begin{pmatrix} \text{surface area of} \\ \text{the pyramid} \end{pmatrix}$$

where the last equality holds because (area of polygon about A) $= nC^2 \tan(\pi/n)$ and (surface area of the pyramid) $= \frac{1}{2}(2C\tan(\pi/n))D \times n = nCD\tan(\pi/n)$ since a single triangular face of the pyramid has base $2C\tan(\pi/n)$ and height D. Hence

$$\text{(surface area of the pyramid)} = \text{(area of polygon about } B\text{)}.$$

Now

$$\text{(area of polygon about } B\text{)} : \text{(area of polygon in } B\text{)} < S : B$$

by $(*)$, therefore

$$\text{(surface area of the pyramid)} = \text{(area of polygon in } B\text{)} < S : B.$$

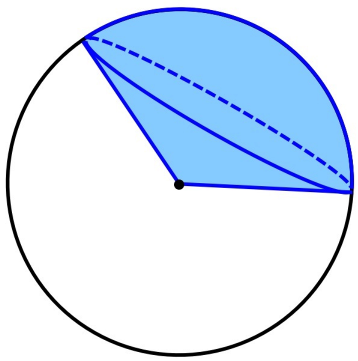
But (surface area of the pyramid) $> S$ (since the cone is inscribed in the pyramid) and (area of polygon in B) $< B$, so that

$$\text{(surface area of the pyramid)} : \text{(area of polygon in } B\text{)} > S : B.$$

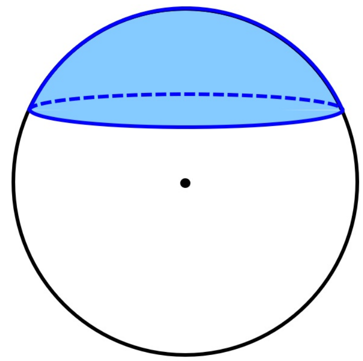
This CONTRADICTION shows that the assumption that $B < S$ is false. Similarly, we can show that the assumption $B > S$ is false (as Archimedes does), concluding the proof that $B = S$. ∎

Several propositions are then given concerning cutting cones and solid rhombi with a plane (in Propositions 16–20). These are then used (along with revolved polygons) to estimate surface areas and volumes of spheres in Propositions 21–32. Propositions 33 and 34 deal with claims (1), (3), and (4) at the beginning of this note. Propositions 35–41 consider additional similar solids of revolution inscribed and circumscribed in a sphere, and the ratios between their volumes and surface areas.

Propositions 42 and 43 deal with claim (2) at the beginning of this note. Proposition 44, the final claim in Book I, concerns the volume of a spherical sector (that is, the volume that results from joining the base of a spherical segment with altitude less than the radius of the circle to the center of the sphere; see the figure below).



A spherical sector (the volume of the sphere contained in the shaded cone and shaded spherical segment)



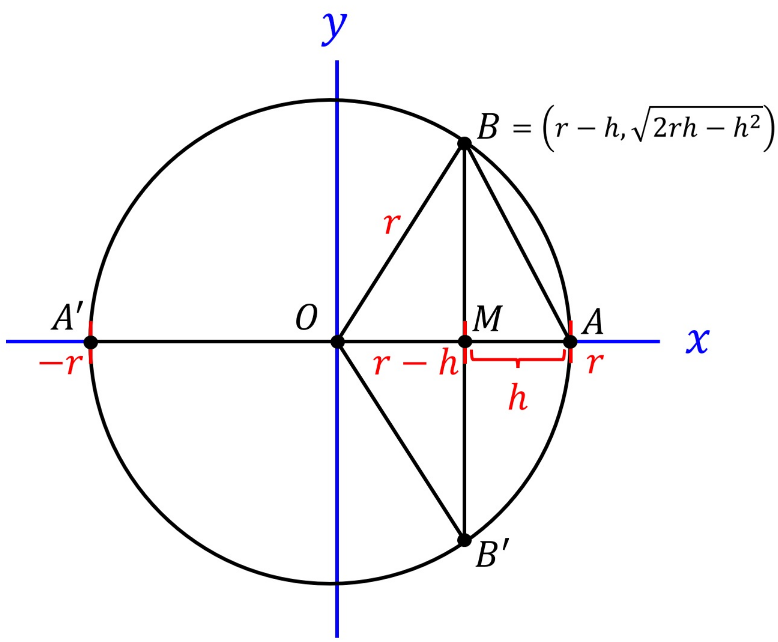
A spherical segment with altitude less than the radius of the sphere

Proposition 44 claims that the volume of such a spherical sector is equal to the volume of a cone with a base of area equal to the area of the spherical segment (which is given by Propositions 42 and 43) and height equal to the radius of the sphere.

Note AW.E. *Sphere and Cylinder* Book II has six problems and three theorems (though Archimedes states these as nine propositions; Propositions 2, 8, and 9 are the theorems). Notice that a spherical sector (see the figure above) can be expressed as the union of a spherical segment (also called a "spherical section") and a cone. Hence, the volume of the segment can be found by subtracting the cone from the sector. We use this idea in our proof of Archimedes' Proposition 2. We paraphrase Proposition 2 and its proof (based on Heath's *History Volume 2*, page 42) as follows.

Proposition 2. A spherical segment of a sphere of radius r with altitude h has a volume equal to a cone with the same base as the spherical segment and height $h(3r-h)/(2r-h)$.

Proof. As discussed above, a segment of a sphere can be found by subtracting a cone from a sector. We can find the volume of the segment of the sphere BAB' by subtracting the volume of the cylinder OBB' from the sector as $OBAB'$.



By Proposition 44 of Book I, the volume of a cone with a base of area equal to the area of the spherical segment and height equal to the radius of the sphere. The surface of a segment of a sphere is equal to a circle the radius of which is equal to the straight line drawn from the vertex of the segment to a point on the circumference of the base of the segment by (2) of Note AW.D. So the volume of sector in the figure above is

$$V = \frac{1}{3}\pi(AB)^2 r - \frac{1}{3}\pi(BM)^2(OM),$$

where we know the volume of the cone from Euclid's *Elements*, Book XII, Proposition 10. We introduce a coordinate system, as in the figure. The equation of the circle is $x^2 + y^2 = r^2$, so with the x-coordinate of B as $r - h$ we have the y-coordinate as $y = \sqrt{r^2 - (r-h)^2} = \sqrt{2rh - h^2}$. Then

$$(AB)^2 = ((r-h) - r)^2 + (\sqrt{2rh - h^2} - 0)^2 = h^2 + 2rh - h^2 = 2rh,$$

$(BM)^2 = 2rh - h^2$, $AA' = 2r$, and $A'M = 2r - h$. Therefore,

$$(AB)^2 : (BM)^2 = (2rh) : (2rh - h^2) = (2r) : (2r - h) = (AA') : (A'M).$$

We now have the volume of the segment is

$$\begin{aligned} V &= \frac{1}{3}\pi(AB)^2 r - \frac{1}{3}\pi(BM)^2(OM) = \frac{1}{3}\pi(BM)^2\left(\frac{(AB)^2}{(BM)^2}r - (OM)\right) \\ &= \frac{1}{3}\pi(BM)^2\left(\frac{2r}{2r-h}r - (r-h)\right) = \frac{1}{3}\pi(BM)^2\left(\frac{2r^2}{2r-h} - \frac{2r^2 - 3rh + h^2}{2r-h}\right) \\ &= \frac{1}{3}(BM)^2\left(\frac{3rh - h^2}{2r-h}\right) = \frac{1}{3}(BM)^2\left(\frac{h(3r-h)}{2r-h}\right). \end{aligned}$$

This is the volume of a cone with the same base as the spherical segment (namely, a circle with radius (BM)) and height $h(3r - h)/(2r - h)$. ∎

Book II Proposition 4 ("the most important proposition in the Book," according to Heath in *History, Volume 2*, page 43) states the problem:

Proposition 4. To cut a given sphere by a plane so that the volumes of the segments are to one another in a given ratio.

In Eves' Problem Study 6.2 part (e), it is to be shown that with r as the radius of the sphere, x the ("unknown") distance of the cutting plane from the center of the sphere, and m/n the given ratio, we have the cubic equation $n(r - x)^2(2r + x) = m(r+x)^2(2r-x)$. Setting the altitude of the smaller segment to h we have $h = r-x$, so that $2r + x = 3r - h$, $r + x = 2r - h$, and $2r - x = r + h$. Converting the above cubic in terms of unknown x into a cubic in terms of unknown h we have $nh^2(3r - h) = m(2r - h)(r + h)$ or

$$3nh^2r - nh^3 = m(4r^2 - 4hr + h^2)(r_h) = m(4r^3 - 4hr^2 + h^2r + 4hr^2 - 4h^2r + h^3)$$

or $(-n - m)h^3 + (3n + 3m)h^2R - 4mr^3 = 0$ or

$$h^3 - 3h^2r + \frac{4m}{n+m}r^3 = 0 \qquad (*)$$

(see Heath's *History, Volume 2*, page 43). As opposed to solving this particular equation, Archimedes considers a more general relationship of the form $(r + h) : b = c^2 : (2r - h)^2$ where b is some given length (so b is some multiple of the given constant radius r) and c^2 is some given area (so c^2 is some multiple of r^2; Archimedes seems to be conscious of the units of length, area, and volume here). The relationship gives

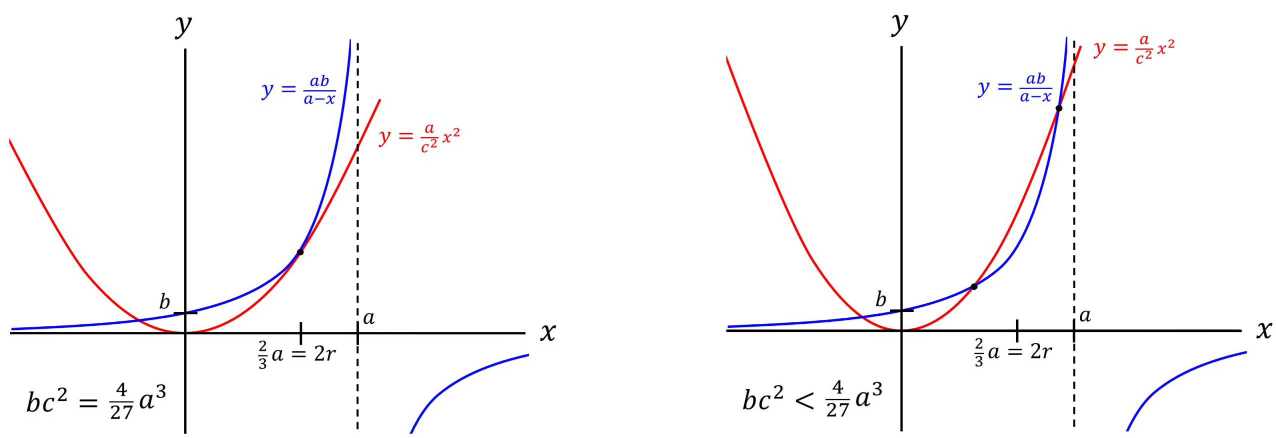
$$\frac{r+h}{b} = \frac{c^2}{(2r-h)^2} \text{ or } (r + h)(2r - h)^2 = bc^2 \qquad (**)$$

or $(r + h)(4r^2 - 4hr + h^2) = bc^2$ or $4r^3 - 4hr^2 + h^2r + 4r^2h - 4h^2r + h^3 = bc^2$ or $h^3 - 3h^2r + 4r^3 = bc^2$. With $bc^2 = \left(4 - \frac{4m}{n+m}\right) r^3 = \frac{4n}{n+m}r^3$, this reduces to $(*)$. Archimedes makes an additional substitution, $x = 2r - h$ and $a = 3r$ to produce from $(**)$ the cubic

$$(r + h)(2r - h)^2 = bc^2 \text{ or } (a - x)x^2 = bc^2 \qquad (***)$$

(this x is not the same as the x used in Eves' equation given in Problem Study 6.2(e)). Archimedes states that he will analyze and solve this problem "at the end" of Book II. However, this does not appear in the extant versions of Book

II. However, Eutocius (circa 480–circa 540) "gives solutions taken from 'an old book' which he managed to discover after a laborious search . . . [and which] he with fair reason assumed to contain the missing *addendum* by Archimedes" (Heath, *History, Volume 2*, page 45). Heath refers to this analysis as "Archimedes's own solution of the cubic." Set $x^2 = \frac{c^2}{a}y$ where $y = \frac{ab}{a-x}$ so that $x^2 = \frac{c^2}{a}\left(\frac{ab}{a-x}\right)$ or $x^2(z-x) = bc^2$, which matches (∗∗∗). A solution to the cubic would be a given by the x-coordinate of a point (x, y) on both the parabola $y = \frac{a}{c^2}x^2$ an on the hyperbola $= \frac{ab}{a-x}$ (with vertical asymptote of $x = a$ and horizontal asymptote of $y = 0$; see the figure below).



Archimedes is only interested in positive solutions to the cubic and so ignores the negative solution. He shows that if $bc^2 = \frac{4}{27}a^3 = \frac{4}{27}(3r^3) = 4r^3$ then there is a unique (positive) solution at $x = \frac{2}{3}a = 2r$ (see the figure above, left). In the spherical segment problem this corresponds to one segment of altitude $x = 2r - h = 0$ and the other segment of altitude $h = 2r$; this is the case where the ratio is 0, a case that Archimedes would not consider since '0' was not recognized as a number in his time. Archimedes also shows that the cubic has two positive solutions if $bc^2 < \frac{4}{27}a^3 = 4r^3$ (as seen in the figure above right), one solution is greater than $\frac{2}{3}a = 2r$ which is meaningless for the segment problem, and one is between 0 and $\frac{2}{3}a = 2r$ which is the desired solution of the segment problem. In the spherical segment problem, we have $bc^2 = \frac{n}{n+m} 4r^3 < 4r^3 = \frac{4}{27} a^3$ (see above), so a single positive, meaningful solution exists. In this way, Archimedes has shown the existence of a solution and has given a constructive way to find the solution (in terms of intersecting parabolas and hyperbolas, not in terms of compass and straight edge, but still a constructive solution). In Proposition 8, it is proved that for a sphere cut by a plane, with S and V as the surface area and volume, respectively, of the larger segment, and S' and V' as the surface area and volume, respectively, of the smaller segment, we have $V : V' < S^2 : (S')^2$ but $V : V' > S^{3/2} : (S')^{3/2}$. In the final proposition of Book II, Proposition 9, it is proved that of all segments of spheres which have their surfaces equal, the hemisphere is of the greatest in volume. That is, if we consider segments of spheres of different radii, all the segments of which having the same surface area S, then the segment with largest volume will be a spherical segment of a sphere with total surface area equal to $2S$ (and a hemisphere of area S, so that $S/2 = 4\pi r^2$ and $r = \sqrt{S/(8\pi)}$, because the surface area of a sphere is $A = 4\pi r^2$). Archimedes proof is based on considering spherical segments with the given surface area which are more than a hemisphere (in which case the radius must be less than $\sqrt{S/(8\pi)}$) and spherical segments with the given surface area less than a hemisphere (in which case the radius must be more that $\sqrt{S/(8\pi)}$).

Revised: 4/28/2024