Supplement. The Content of Archimedes' Work, Part 2

Note. We continue surveying Archimedes' works. In *A History of Greek Mathematics, Volume 2*, Heath gives the chronological listing of Archimedes' works as:

1. *Quadrature of the Parabola*
2. *Sphere and Cylinder* Book I
3. *Sphere and Cylinder* Book II
4. *Spiral Lines*
5. *Conoids and Spheroids*
6. *The Method*
7. *On Floating Bodies*
8. *On Balancing Planes*

We start with *Spiral Lines*.

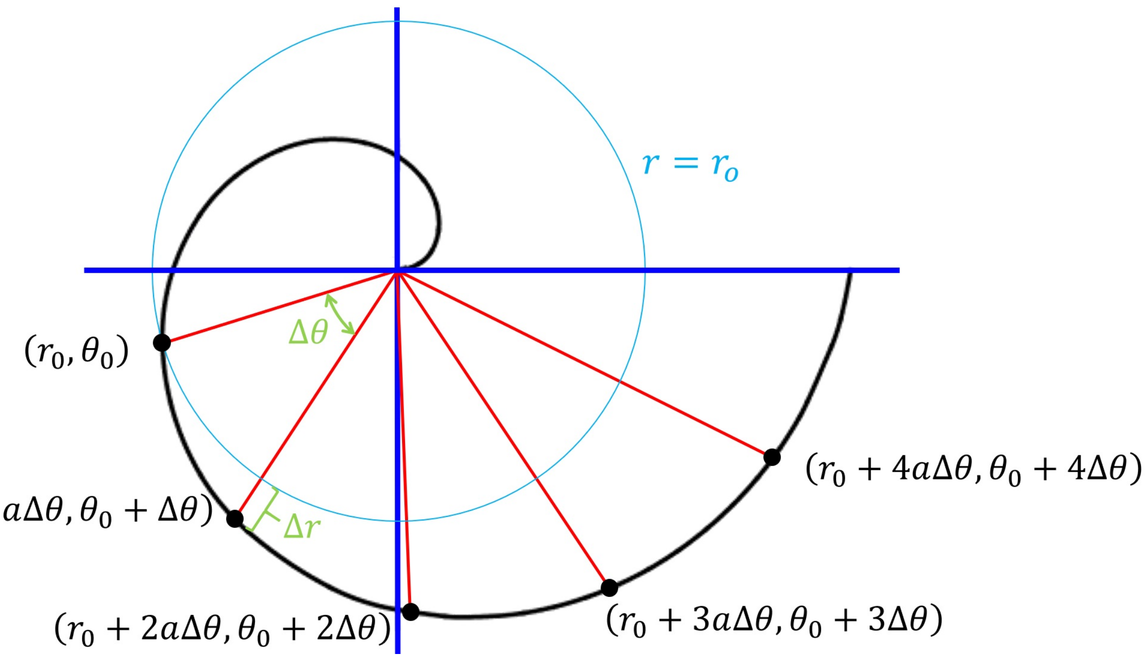
Note AW2.A. Some of the content of *Spiral Lines* is briefly described in the history part of Introduction to Modern Geometry (MATH 4157/5157); see my online notes for this on Section 4.2. The Archimedean Spiral. After several preliminary results (Propositions 1 to 11), Archimedes gives the following definition of a spiral. The statements of definitions and propositions given in this note are base on Thomas Heaths *The Works of Archimedes* (Cambridge University Press, 1897).

Definition. If a straight line drawn in a plane revolve at a uniform rate about one extremity which remains fixed and return to the position from which it started, and it, at the same time as the line revolves, a point move at a uniform rate along the straight line beginning from the extremity which remains fixed, the point will describe a *spiral* in the plane.

The references to revolving a straight line (segment), a moving point, and uniform rates are not exactly clean, clear mathematical definitions. In Archimedes defense, he does not have access to coordinate systems (and, as we have seen, he does not shy away from the use of physical ideas). In modern notation, we can express a (Archimedean) spiral in polar coordinates as $r = a\theta$ (where a determines the "uniform rates"). After the statement of the definitions, Propositions 12, 14, and 15 give the fundamental properties of the spiral as related to the distance from the origin and the angle through which the revolving line has swept. Proposition 13 concerns tangent lines to the spiral.

Proposition 12. If any number of straight lines drawn from the origin to meet the spiral make equal angles with one another, the [lengths of the] lines will be in arithmetical progression.

Archimedes apparently doesn't provide a proof and Heath simply adds a parenthetic comment that "The proof is obvious." The idea is that as the line segment rotates uniformly it sweeps out equal angles in equal "times," the uniformly moving point creates line (segments) of proportional lengths. In terms of polar coordinates we have $r = a\theta$, so that if θ changes from θ_0 to $\theta_0 + \Delta\theta$ and then to $\theta_0 + 2\Delta\theta$, then r changes from r_0 (say) to $r_0 + \Delta r$ and then to $r_0 + 2\Delta r$ (and so forth) where $\Delta r = a\Delta\theta$. So when we add n multiples of $\Delta\theta$ to θ_0, the corresponding line segments are of lengths $r_0 + a\Delta\theta$, $r_0 + 2a\Delta\theta$, $\ldots$, $r_0 + na\Delta\theta$. That is, with $\Delta\theta$ a constant (which gives "angles equal to one another") then the lengths of the line segments follow this arithmetic progression. See the next figure.



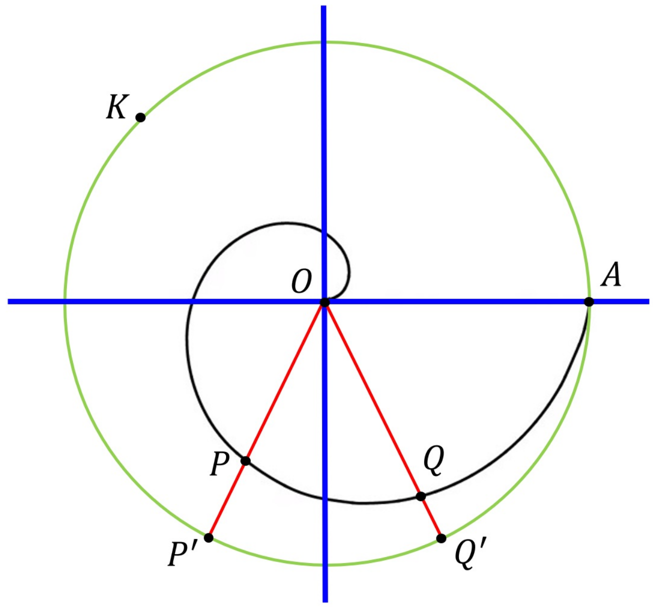
Proposition 13. If a straight line touch the spiral, it will touch it in one point only.

The expression "straight line touch the spiral" indicates that the line is tangent to the spiral. So Proposition 13 demonstrates that that a line tangent to the Archimedean spiral cannot be tangent at a second point.

Proposition 14. If O be the origin, and P, Q two points on the first turn of the spiral, and if OP, OQ produced meet the 'first circle' $AKP'Q'$ in P', Q' respectively, OQ being the initial line, then

$$OP : OQ = (\text{arc } AKP') : (\text{arc } AKQ').$$

See the figure below. Proposition 14 is unsurprising since, in polar coordinates, the lengths of OP and OQ are proportional to the angles determining them (when $r = a\theta$, the constant of proportionality is a), and the lengths of arcs on the circle are also proportional to the angle determining them (with constant of proportionality equal to the radius of the circle when θ is measured in radians).



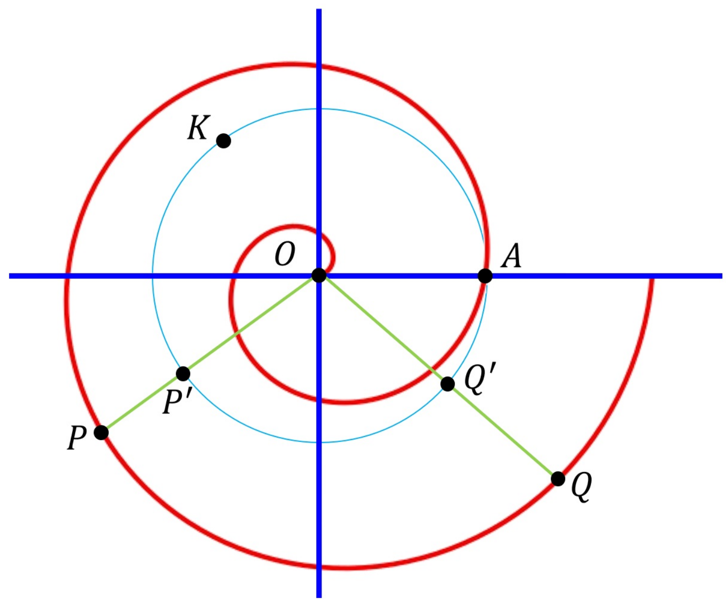
Before Proposition 12, other definitions are given that are needed for the next proposition.

Definition. Let the length which the point that moves along the straight line describes in one revolution be called the *first distance*, that which the same point describes in the second revolution the *second distance*, and similarly let the distances described in further revolutions be called after the number of the particular revolutions.

Definition. Let the circle drawn with the origin as center and the first distance as radius be called the *first circle*, that drawn with the same center and twice the radius the *second circle*, and similarly for the succeeding circles.

Proposition 15. If P, Q be points on the second turn of the spiral, and OP, OQ meet the 'first circle' $AKP'Q'$ in P', Q', as in the last proposition, and if c be the circumference of the first circle, then

$$OP : OQ = (c + (\text{arc } AKP')) : (c + (\text{arc } AKQ')).$$



Proposition 15 is also unsurprising and simply extends Proposition 14 from the first revolution of the straight line to the second revolution. In terms of polar coordinates, it also follows based on the proportionality of the lengths of OP and OQ to the angles determining them and the proportionality of the lengths of the arcs to the first distance (that is, in our new terminology, the radius of the first circle). This can be extended from the second turn to the nth turn of the spiral to get

$$OP : OQ = ((n-1)c + (\text{arc } AKP')) : ((n-1)c + (\text{arc } AKQ')).$$

Archimedes states this as a corollary of Proposition 15. We need another definition before stating describing the next propositions.

Definition. If from the origin of the spiral any straight line be drawn, let that side of it which is in the same direction as that of the revolution be called *forward*, and that which is in the other direction *backward*.

The figures presented above all have counter clockwise as the forward direction, as would be the convention in polar coordinates where counter clockwise is the "positive" direction of θ. However, all of the figures in Heath's *History, Volume 2* and his *The Works of Archimedes* which relate to *Spiral Lines*, give the positive direction as clockwise. In Propositions 16 and 17, Archimedes proved that the angle made by the tangent at a point with the radius "vector" to that point (i.e., the line segment from the origin to that point) is obtuse on the forward side of the radius vector, and acute on the backward side of the radius vector. Propositions 18, 19, and 20 concern the points of intersection of a tangent to the spiral with a given radius vector and a perpendicular to the radius vector (the line segment resulting is called a "subtangent"; it is a part of the tangent line). The rest of the proposition, Propositions 21 to 28, concern areas or portions of the spiral. In particular, Propositions 21, 22, and 23 concern approximating areas bounded by the spiral with sectors of circles centered at the origin. Proposition 24 proves that the area bounded by the first turn of the spiral is $\dfrac{1}{3}\pi(2\pi a)^2$ where, in polar coordinates, the spiral is $r = a\theta$. Archimedes states this as:

Proposition 24. The area bounded by the first turn of the spiral and the initial line is equal to one-third of the 'first circle'.

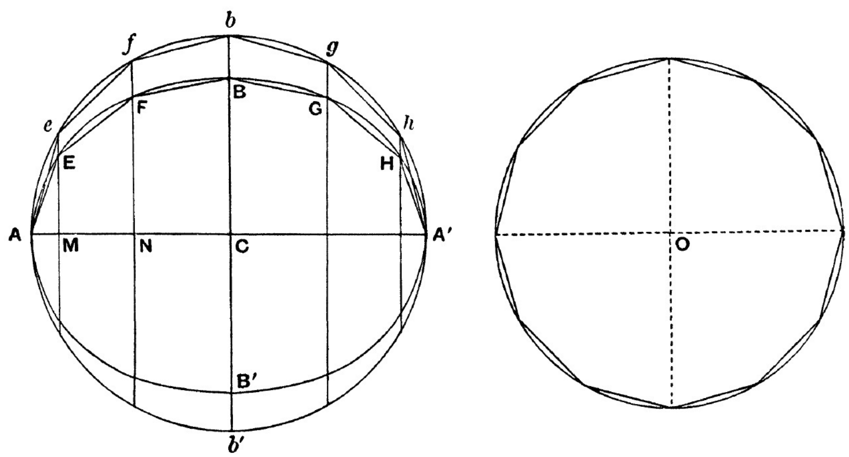
Notice that with $r = a\theta$, the first distance is $2\pi a$. This is the radius of the first circle, so the first circle has area $\pi(2\pi a)^2$ and we see that the formula given above agrees with Archimedes statement. This is easily demonstrated by integrating in polar coordinates. Such a solution is given in the history part of Introduction to

Modern Geometry (MATH 4157/5157); see Section 4.2. The Archimedean Spiral and notice the proof given in the Beamer supplement to this section. Proposition 25 concerns the area bounded by the second turn of the spiral, and Proposition 26 concerns the area of a section of the spiral that is less than one complete turn.

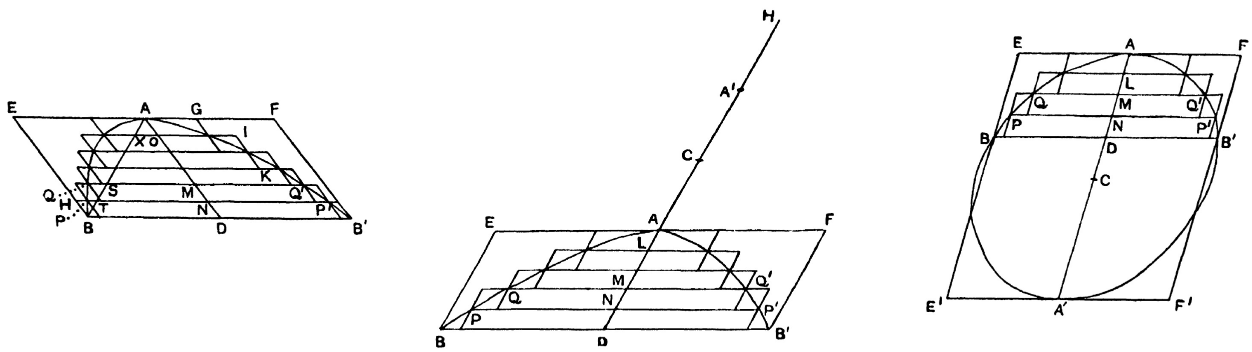
Note AW2.B. *Conoids and Spheroids* primarily concerns volumes of revolution of conic sections. Archimedes considers both right sections (where the volume of revolution is cut by a plane perpendicular to the axis of the conic) and oblique sections (where the cutting plane is not perpendicular to the axis of the cone). After stating definitions, Archimedes gives two preliminary lemmas and 32 propositions. The definitions, in modern terminology, include (1) the *right-angled conoid* (or paraboloid of revolution), (2) the *obtuse-angled conoid* (or hyperboloid of revolution), (3) the *spheroids* that are it obling (or the revolution of an ellipse about its major axis) and *flat* (or the revolution of an ellipse about its minor axis). The two lemmas and Proposition 1 and 2 are used in proving later propositions. Propositions 4, 5, and 6 involve finding the area of an ellipse. Proposition 4 gives the most direct result (stated next), and Propositions 5 and 6 give the area as it relates to other objects (circles and rectangles).

Proposition 4. The area of any ellipse is to that of the auxiliary circle [a circle with radius equal to the major axis of the ellipse] as the minor axis to the major.

With the lengths of the major axis and minor axis as a and b, respectively, we have the area of the auxiliary circle is πa^2 so, with A as the area of the ellipse, Proposition 4 implies $A : \pi a^2 = b : a$, or $A = \pi a^2 b/a = \pi ab$, as expected. Archimedes gives a standard method of exhaustion proof. Assuming that O is a circle satisfying the desired ratio (that is, the area of circle O is to the area of the auxiliary circle as the minor axis of the ellipse is to the major axis of the ellipse), Archimedes first assumes that the area of O is greater than the area of the ellipse. Then he inscribes a regular $4n$-gon inside circle O such that its area is greater than that of the ellipse (which can be done by Proposition 6 of *Sphere and Cylinder* Book I). See the figure below, left.



He then inscribes a similar polygon in the auxiliary circle (in the figure, right). He shows the areas of the two polygons are the same, getting a contradiction to the assumption that the area of O is greater that of the ellipse. When assuming the area of O is less than that of the ellipse, he inscribes a polygon in the ellipse (also in the figure left) and gets a similar contradiction. Propositions 7 and 8 consider constructions of (possible oblique) cones that have a given ellipse as a cross section. Proposition 9 considers a similar problem for cylinders (instead of cones). Propositions 11 to 18 give properties of conoids and spheroids which are derived from the generating conics. These results often concerning cross sections that result when the solid is "cut by a plane." Proposition 19 gives conditions which will allow for the use of the method of exhaustion in addressing volumes in the rest of *Conoids and Spheroids.* It shows that, for the solids under consideration, a collection of cylinders and 'frustra of cylinders' (i.e., cylinders truncated by one or two planes oblique to the axis of the cylinder; the cutting planes will be parallel in this application) can be inscribed in or circumscribed around the solid such that the difference in volume between the solid under consideration and the collection of solids is as small as we please.



From Heath's *History of Greek Mathematics, Volume 2*, page 59. The figures are cross sections of a (left to right) paraboloid, hyperboloid, and spheroid.

In the figure above, the plane of the page contains the axis AD of the conoid or spheroid. This plane intersects the base of the solid along line segment BB'; the base is then a circle or ellipse depending on the angle (right angle or non-right angle) the base makes with the axis. The horizontal lines are evenly spaced along the axis of each solid. The parallelograms in the figure each correspond to inscribed or circumscribed frustra of a cylinder. Propositions 20, 21, and 22 lead to the conclusion (quoting from Heath's *Works of Archimedes*, page 131):

Propositions 21, 22. Any segment of a paraboloid of revolution is half as large again as the cone or segment of a cone which has the same base and the same axis.

As expected, this is proved using the method of exhaustion. Notice that requiring the segment of the paraboloid and the segment of the cone to have the same base, Archimedes avoids two separate cases of a cutting plane at right angles to the axis and at non-right angles to the axis. Propositions 23 and 24 compare the volumes of of two segments of the same paraboloid determined by two different cutting planes; the comparisons depending on the length of the axis AD (see figure above, left). Propositions 25 and 26 concern the volume of a segment of a hyperboloid. The volume is given in terms of the volume of a cone with the same base (similar to the case of the paraboloid), but also involves the semimajor axis of the hyperbola. Archimedes calls the semimajor axis the "semidiameter" and introduces it as distance CA where A is the vertex of the hyperbola (remember, the Greeks only considered one connected component of the hyperbola). This is stated in Heath's *Works of Archimedes* (page 136) as: "Let C be the centre of the hyperboloid (or the vertex of the enveloping cone)." Propositions 25 and 26 are stated as (see the figure above, middle):

Propositions 25, 26. In any hyperboloid of revolution, if A be the vertex and AD the axis of any segment cur by a plane, and if CA be the semidiameter of the hyperboloid through A (CA being of course in the same straight line with AD), then

$$\text{(segment)} : \left(\begin{array}{c}\text{cone with same}\\ \text{base and axis}\end{array}\right) = (AD + 3CA) : (AD + 2CA).$$

Again, the method of exhaustion is used (as it will be for the spheroid). Propositions 27, 28, 29, and 30 concern volumes of spheroids (remember, "spheroids" are revolutions of an *ellipse* about its minor or major axis, and is not to be confused with segments of a sphere). As in the two previous cases, AD denotes the length of the axis of the segment, but this time CA is the length of the semi-axis about which the ellipse is revolved. These state (again, quoting Heath's *Works*; see the figure above, right):

Propositions 27, 28, 29, 30. (1) In any spheroid whose centre is C, if a plane meeting the axis cut off a segment not greater than half the spheroid and having A for is vertex and AD for is axis, and if $A'D$ by the axis of the remaining segment of the spheroid, then

$$\text{(first segment)} : \left(\begin{array}{c} \text{cone or segment of cone} \\ \text{with same base and axis} \end{array} \right) = (CA + A'D) : (A'D)$$
$$[= (3CA - AD : 2CA - AD)].$$

(2) As a particular case, if the plane passes though the centre, so that the segment is half the spheroid, half the spheroid is double of the cone or segment of a cone which has the same vertex and axis.

The last two results in *Conoids and Spheroids*, Propositions 31 and 32, give the ratio of the volumes of the two (unequal) segments that result when cutting a spheroid with a plane. As a final observation, all of the volume results given in this work of Archimedes can be dealt with today using with Calculus 2 in volumes of revolution (MATH 1920; see my online Calculus 2 notes on Section 6.1. Volumes Using Cross-Sections and Section 6.2. Volumes Using Cylindrical Shells) or in Calculus 3 with double integrals (MATH 2110; see my online Calculus 3 notes on Section 15.2. Double Integrals over General Regions).

Note AW2.C. Archimedes' *On Floating Bodies* Book I gives the foundations of hydrostatics. He starts with a postulate. As stated it Heath's *History, Volume 2* (page 92):

Postulate 1. Let it be assumed that a fluid is of such a nature that, of the parts of it which lie evenly and are continuous, that which is pressed the less is driven along by that which is pressed the more; and each of its parts is pressed by the fluid which is perpendicularly above it except when the fluid is shut up in anything and pressed by something else.

The propositions of Book I largely deal with buoyancy. The arguments are, in the opinion of your humble instructor, less mathematical than Archimedes' other work. The concept of the pressure of the fluid is often invoked in the proofs. In this note, we don't explore the proofs in detail. Some of the propositions are (quoting from Heath's *Works*):

Proposition 3. Of solids those which, size for size, are of equal weights with a fluid will, if let down into the fluid, be immersed so that they do not project above the surface but do not sink lower.

Proposition 4. A solid lighter than a fluid will, if immersed in it, not be completely submerged, but part of it will project above the surface.

Proposition 5. Any solid lighter than a fluid will, if placed in the fluid, be so far immersed that the weight of the solid will be equal to the weight of the fluid displaced.

Proposition 6. If a solid lighter than a fluid be forcibly immersed in it, the solid will be driven upwards by a force equal to the difference between its weight and the weight of the fluid displaced.

Notice that Archimedes is close to the ideas from Newtonian mechanics of treating weight as a force (a vector quantity in the downward direction) and the buoyancy as a force (a vector quantity in the upward direction).

Proposition 7. A solid heavier than a fluid will, if placed in it, descend to the bottom of the fluid, and the solid will, when weighed in the fluid, be lighter than its true weight by the weight of the fluid displaced.

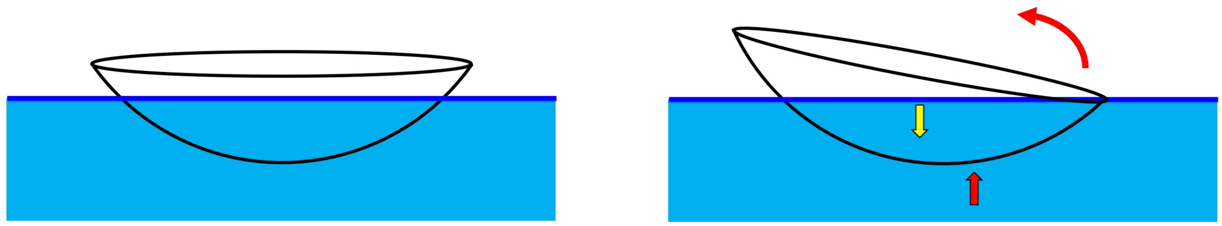
Eves' in Problem Study 6.3 of his *An Introduction to the History of Mathematics*, 6th edition, paraphrases Proposition 7 as (see page 188): "A body immersed in a fluid is buoyed up by a force equal to the weight of the displaced fluid." In his *Works of Archimedes*, Heath gives what he thinks is the process used by Archimedes in solving "The Crown Problem" (see Note 6.2.C of Section 6.2. Archimedes). Heath states (see his page 259): "This proposition [Proposition 7] may, I think, safely be regarded as decisive of the question how Archimedes determined the proportions of gold and silver contained in the famous crown... The proposition suggests in fact the following method." In Eves' Problem Study 6.3 sets up the question in terms of the following variables.

(a) Let a crown of weight w pounds be made of of w_1 pounds of gold and w_2 pounds of silver. Suppose that w pounds of pure gold loses f_1 pounds when weighed in water, that w pounds of pure silver loses f_2 pounds when weighed in water, and that the crown loses f pounds when weighed in water. [Weights f_1, f_2, and f are the respective amounts by which the gold, silver, and crown are "bouyed up," as given by Proposition 7.] Show that $\dfrac{w_1}{w_2} = \dfrac{f_2 - f}{f - f_1}$.

(b) Suppose the crown of (a) displaces a volume of v cubic inches when immersed in water, and that lumps of pure gold and pure silver that are of the same weight as the crown displace, respectively, v_1 and v_2 cubic inches when immersed in water. Show that $\dfrac{w_1}{w_2} = \dfrac{v_2 - v}{v - v_1}$.

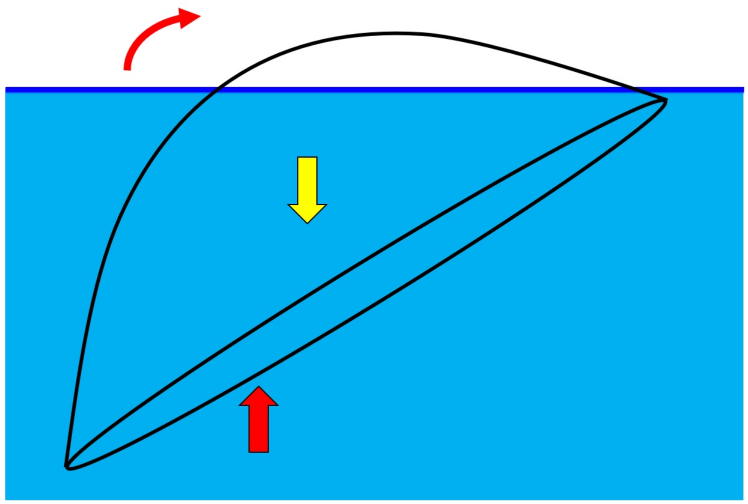
The approach given in (a) "corresponds pretty closely to that described in the poem *de ponderibus et mensuris* (written probably about 500 A.D." [Heath's *Works of Archimedes*, page 260). With it, one could calculate f, f_1, and f_2 by weighing the crown (of known "usual" weight w), a piece of pure gold (of known "usual" weight w), and a piece of pure silver (of known "usual" weight w) in water. Then the ratio w_1/w_2 can be calculated indicating the ratio of gold to silver in the crown. The approach given in (b) is consistent with the account given by Vitruvius (circa 75 BCE–circa 10 BCE) in *De architect*, Book IX. With this, Archimedes could calculate the ratio w_1/w_2 by calculating the *volumes* that the crown (v), an equal weight of gold (v_1), and an equal weight of silver (v_2) displace when submerged. It is easy to see how Vitruvius could extract the bathtub story from this (again, see Note 6.2.C of Section 6.2. Archimedes). After Proposition 7, Archimedes states "Postulate 2" in which de claims that the force of buoyancy on a submerged body is in an upward direction through the center of gravity of the body. Book I ends with Propositions 8 and 9 which consider a segment of a sphere which is lighter than a fluid in which it is immersed. These two propositions also address the stability of such a floating body.

Proposition 8. If a solid in the form of a segment of a sphere, and of a substance lighter than a fluid, be immersed in it so that its base does not touch the surface, the solid will rest in such a position that its axis is perpendicular to the surface [see the figure below, left]; and, if the solid be forced into such a position that its base touches the fluid on one side and be then set free, it will not remain in that position but will return to the symmetrical position [figure below, right].



In the figure here (right), the submerged part is buoyed upward through the center of gravity of the submerged part (in red; this is an application of Postulate 2) and the weight acts downward through the center of gravity of the segment (in yellow). This results in the movement of the segment back to its equilibrium position (right). In Book II of *On Floating Bodies*, 10 propositions are given that explore the conditions of stability of a segment of a paraboloid of revolution with a base perpendicular to the axis. Four pairs of propositions (Propositions 2 and 3, 4 and 5, 6 and 7, 8 and 9) cover different cases related to the height of the paraboloid versus the principal parameter p of the generating parabola (or *semi latus rectum* of the parabola; see Note 3.1.A in the history part of my online notes for Introduction to Modern Geometry [MATH 4157/5157] on Section 3.1. The Parabola), and the relative density (or specific gravity) of the solid to the fluid (that is, the ratio of the density of the solid to the density of the fluid). As an example, we state Proposition 3.

Proposition 3. If a right segment of a paraboloid of revolution whose axis is not greater than $\frac{3}{4}p$, and whose specific gravity is less that that of a fluid, be placed in the fluid with is axis inclined at any angle to the vertical, but so that its base is entirely submerged, the solid will not remain in that position but will return to the position in which the axis is vertical.



In the figure here, the submerged part is buoyed upward through the center of gravity of the submerged part (in red; this is an application of Postulate 2) and the weight acts downward through the center of gravity of the segment (in yellow). This results in the movement of the segment back to its equilibrium position where the axis is vertical.

Note AW2.D. *On Balancing Planes* (also known as *On Plane Equilibriums*) consists of two books. Book I starts with seven postulates concerning balancing weights and centers of gravity. These include (as stated in Heath's *Works of Archimedes*; see paged 189 and 190):

Postulate 1. Equal weights at equal distances are in equilibrium, and equal weights at unequal distances are not in equilibrium but incline towards the weight which is at the grater distances.

Postulate 2. If, when weights at certain distances are in equilibrium, something be added to one of the weights, they are not in equilibrium but incline towards that weight to which the weight from which noting was taken.

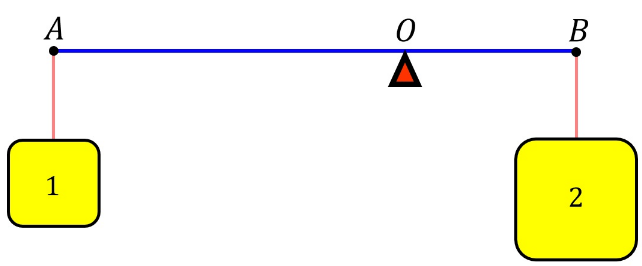
Postulate 6. If magnitudes at certain distances be in equilibrium, (other) magnitudes equal to them will also be in equilibrium at the same distances.

Postulate 7. In any figure whose perimeter is concave in (one and) the same direction the centre of gravity must be within the figure.

Based on the postulates, Archimedes proves 15 propositions. The "greatest hits" are:

Proposition 3. Unequal weights will balance at unequal distances, the greater weight being at the lesser distances.

Propositions 6 and 7. Two magnitudes, whether commensurable or incommensurable, balance at distances reciprocally proportional to the magnitude.



In the figure above, Propositions 6 and 7 imply that $1 : 2 = OB : OA$, where the numbers represent weights. Equivalently, we have $(1)(OA) = (2)(OB)$, as we would expect.

Proposition 10. The centre of gravity of a parallelogram is the point of intersection of its diagonal.

Proposition 14. The centre of gravity of any triangle is at the intersection of the lines drawn from any two angles to the middle points of the opposite sides respectively.

Book II of *On Balancing Planes* has 10 propositions. Propositions 1 through 8 concern the centers of gravity of parabolic segments (not necessarily cut by a line perpendicular to the axis), and Propositions 9 and 10 concern such segments that have been cut a second time with a line parallel to the base of the segment. Archimedes starts with the definition of *inscribed in the recognized manner*, in which a triangle is inscribed in a parabolic segment having the same base and height as the parabolic segment, then such triangles are inscribed in the smaller parabolic segments, and so forth. This is the technique used in Supplement. The Content of Archimedes' Work, Part I in Note AW.C on *Quadrature of the Parabola* (Propositions 18–24) in the proof that a parabolic segment has an area that is 4/3 the area of the triangle inscribed in the segment that has the same base and height as the parabolic segment. We need one more term to understand some of the propositions. The *diameter of a parabolic segment* is a line parallel to the axis of the parabola determining the segment and passing through the midpoint of the base of the segment. Some of the propositions of Book II are:

Proposition 4. The centre of gravity of any parabolic segment cut off by a straight line lies on the diameter of the segment.

Proposition 8. If AO be the diameter of a parabolic segment, and G its centre of gravity, then $AG = \frac{3}{2}GO$.

Proposition 8 gives the precise location of center of gravity of a parabolic segment in a way that it could easily be constructed. Proposition 10 similarly gives the center of gravity of a segment that has been cut a second time with a line parallel to the base of the segment. However, the expression of this point is rather more complicated. In Calculus 2 (MATH 1910), centers of gravity (or "centers of mass") of regions in the plane are computed using integration. See my online Calculus 2 notes on Section 6.6. Moments and Centers of Mass where the center of gravity is computed from the moments about the x-axis, y-axis, and total mass. In Exercise AW2.1, you are asked to use the techniques of Calculus 2 to confirm Proposition 8 for a specific given parabolic segment.

Note. We now turn to some other works of Archimedes. These are either lesser works of Archimedes (like *The Sand Reckoner*), works that are only known from fragments, or are known from secondary sources (as is the case of Archimedes' work on semi-regular polyhedra which is only known from comments from Pappus).

Note AW2.E. *The Sand Reckoner* is "a document of the first importance historically" (Heath's *History, Volume 2*, page 81). This is because Archimedes states that Aristarchus of Samos (circa 310 BCE–circa 230 BCE) considered a sun-centered universe with the Earth and other planets orbiting around it, and the Earth rotating on its axis once a day. This idea was not widely circulated until it was posed by Nicolaus Copernicus (February 19, 1473–May 24, 1543) in his *On the Revolutions of the Celestial Spheres*, which was published shortly before his death. Though Aristarchus' work does not survived, it is mentioned in Aristotle's *The Sand Reck-*

oner. Another work of Aristarchus does survive, *On the Sizes and Distances of the Sun and Moon*. This survives in both Greek and Arabic. It is included in Pappus' (circa 290 CE–circa 350 CE) *Synagoge* on astronomy; Pappus' work is intended as an introduction to astronomy that prepares one to read Ptolemy's (circa 85 CE–circa 165 CE) *Syntaxis Mathematica* on mathematical astronomy, which at the time involved the study of the motion of the sun, moon, and planets in the night sky. Archimedes states in *The Sand Reckoner* (see Heath's *Works of Archimedes*, page 222):

> "His [Aristarchus] hypotheses are that the fixed stars and the sun remain unmoved, that the earth revolves about the sun in the circumference of a circle, the sun lying in the middle of the circumference of a circle, the sun lying in the middle of the orbit, and that the sphere of the fixed stars, situated about the same centre as the sun, is so great that the circle in which he supposes the earth to revolve bears such a proportion to the distance of the fixed starts as the centre of the sphere bears to it surface."

Archimedes' plan in *The Sand Reckoner* is to calculate the number of grains of sand needed to fill the universe. He makes several assumptions about the size of the sun and moon, and the distance to the sphere of the fixed stars. The accuracy (or inaccuracy) of these assumptions is not of much interest. What *is* of interest is the technique of enumeration that Archimedes describes. He defines *orders* and *periods* of numbers (which really just represents exponentiating exponentials). Since in his time, a *myriad* represents 10,000 and there are "traditional names" for numbers up to a myriad, then numbers up to a myriad myriad (or 100,000,000) can be expressed. Such numbers he calls numbers of the *first order*. The number 100,000,000 is the unit of the *second order*, and numbers from 100,000,000 to $(100{,}000{,}000)^2$ are of the *second order*. He then defines numbers of the *third order* as those between $(100{,}000{,}000)^2$ and $(100{,}000{,}000)^3$, and continues to process up to numbers of the 100,000,000th order, ending with $(100{,}000{,}000)^{100{,}000{,}000}$ which we define as P; notice that $P = (10^8)^{100{,}000{,}000} = 10^{800{,}000{,}000}$ so that it has (in our notation) 800,000,000 digits. Archimedes then shifts to classifying numbers from 1 to P as *first period* numbers. He then defines the *first order of the second period* numbers as those between P and $100{,}000{,}000P$, and so forth. He ends with the 100,000,000th order of the 100,000,000th period and the number $P^{100{,}000{,}000}$. Notice that

$$P^{100{,}000{,}000} = P^{10^8} = \left(10^{800{,}000{,}000}\right)^{10^8} = 10^{8\times 10^{16}}.$$

That is, Archimedes can use this scheme to represent numbers with up to 80,000 million million (i.e., 8×10^{16}) digits. For the record, Archimedes estimates that, when filled with sand, the universe "would contain a number of grains of sand less than 10,000,000 units of the eighth order of numbers"; that is, $10^{56+7} = 10^{63}$ grains of sand.

Note AW2.F. Archimedes' approach to enumeration in terms of powers of units (such as powers of 10,000, the first order unit, to produce the second order unit 100,000,000, and powers of 100,000,000 to produce the third unit) has descendants in our modern names for various numbers. In the United States, we refer to 1,000,000 as "one million," as is the case in the United Kingdom. However, in the U.S. we refer to 1,000,000,000 as "one billion," whereas in the U.K. this is called "one thousand million." In the U.K., "one billion" is $1{,}000{,}000^2 = 1{,}000{,}000{,}000{,}000$

(one million million, or one million to the second power; that's where the *bi* in *bi*llion comes from in the U.K.). In the U.S., 1,000,000,000,000 is one trillion (so in the U.S., one million times multiples of 1,000 determine the names). In the U.K. one trillion is

$$1{,}000{,}000^3 = 1{,}000{,}000{,}000{,}000{,}000{,}000,$$

where again the *tri* in *tri*llion represents a power of one million. In the U.S., this number is "one quintillion." A document addressing this and some of its history is online the U.K. Parliament website, as Statistical Literacy Guide: What is a Billion? And Other Units (accessed 4/28/2024).

Note AW2.G. Archimedes *Cattle Problem* was discovered in 1773 in a Greek manuscript found in a German library. It is written as a poem and a general solution was not found until 1880. It involves four different colors of bulls and cows, and several pieces of information relating the resulting eight categories of cattle. For example, "the white bulls were equal to a half and a third of the black [bulls] together with the whole of the yellow [bulls]." The given information leads to seven linear equations. The resulting system of equations is consistent, but since there are eight unknowns and only seven equations, it is underdetermined and there are infinitely many solutions; see my online Linear Algebra (MATH 2010) notes on Section 1.4. Solving Systems of Linear Equations and notice Theorem 1.7, "Solutions of $A\vec{x} = \vec{b}$," part (2). Since the unknowns represent numbers of cattle, then the only desired solutions involve positive integers (that is, this is a Diophantine problem; these will be further explored in Section 6.8. Diophantus). The smallest solution involves values of several million for each of the unknowns, and a total number of cattle of 50,389,082; all solutions are integer multiples of this solution. However, Archimedes imposes two other restrictions. He requires the number of black and white bulls to be a perfect square, and the number of dappled and yellow bulls is a triangular number. With these additional restrictions, the smallest solution is roughly $7.76 \times 10^{206,544}$ (as was shown in 1880). All digits of this were printed out by computer for the first time in 1965. The information in this note is based on the Wikipedia page on Archimedes's cattle problem (accessed 4/28/2024).

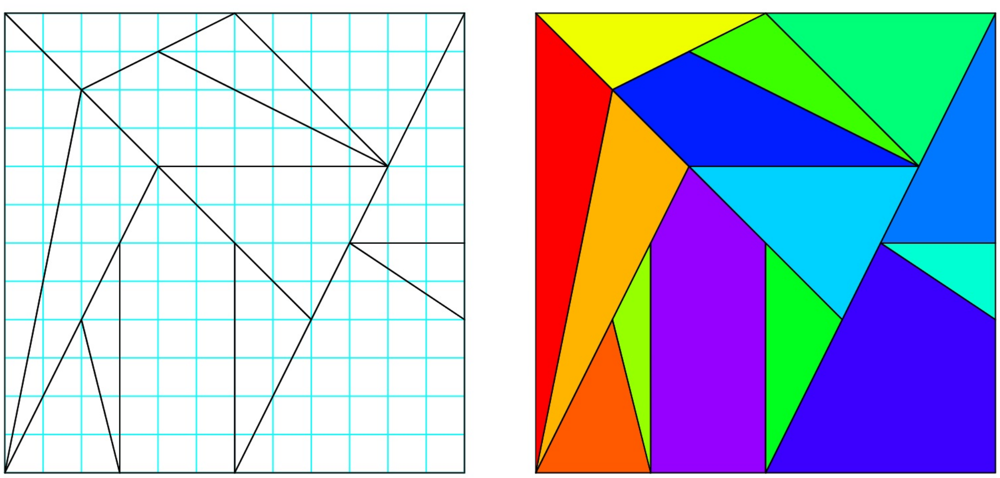
Note AW2.H. Archimedes' *Measurement of a Circle* only contains three propositions. It may be only a fragment of a once-larger work. In Proposition 1, Archimedes proves, by the method of exhaustion, that the area of a circle is equal to that of a right-angled triangle in which the height of the triangle is equal to the radius r of the circle and the base of the triangle is equal to the circumference C of the circle. In other words, $A = \frac{1}{2}Cr$. Recall (see Section 4.8. A Chronology of π, Note 4.8.A) that Euclid's *Elements*, Book XII, Proposition 2 shows that "Circles are to one another as the squares on the diameters." That is, the area of a circle is proportional to the square of its diameter (and hence is proportional to the square of its radius). The constant of proportionality between the area and the square of the radius is how we *define* π. Then we have $A = \pi r^2$, by definition. We now see that Proposition 1 allows us to express the circumference C of a circle in terms of the radius r of the circle constant of proportionality π. We have $A = \pi r^2 = \frac{1}{2}Cr$ or $C = 2\pi r$. Proposition 2 depends on Proposition 3, and "it cannot have been placed by Archimedes before Prop. 3" (Heath's *History, Volume*

2, page 50). This is further reflective of the fact that the version of *Measurement of a Circle* that we have cannot be the original form presented by Archimedes. Proposition 3 is the main result of the work and proves that $3\frac{10}{71} < \pi < 3\frac{1}{7}$. Notice that $3\frac{1}{7} - 3\frac{10}{71} = \frac{71-70}{497} = \frac{1}{497} \approx 0.002$, so this is a good approximation of π (valued to two decimal places). You may have seen $3\frac{1}{7} = 22/7$ used as a nice rational approximation of π. Ultimately, Archimedes inscribes a regular 96-gon and circumscribes a regular 96-gon around a circle. He uses this to get an upper and lower bound on the area of the circle, from which he can estimate π. He starts with an approximation of $\sqrt{3}$. Without any explanation, he claims that $\frac{265}{153} < \sqrt{3} < \frac{1351}{780}$. This is easily (though tediously) verified, but it has been a source of fascination among historians of mathematics as to the technique used by Archimedes; see Heath's *History, Volume 2*, pages 51 and 52 for more on this. Archimedes starts with a right triangle and a 30° angle, so $\sqrt{3}$ arises early in his computation (see slides 54 and 55 or the PowerPoint presentation mentioned below). Archimedes also has to make other approximations to square roots, since he deals with right triangles in his constructions of the regular polygons. For example, he shows that $\sqrt{\frac{349450}{23409}} > \frac{591\frac{1}{8}}{153}$, based on the fact that $349450 > (591\frac{1}{8})^2$. Again, he gives no hints as to how he came to make these specific approximations. Details of the specific computations for the circumscribed 96-gon (and an upper bound on π) are given in Supplement. Archimedes: 2,000 Years Ahead of His Time (in PowerPoint; see slides 52 to 86).

Note AW2.I. In the Archimedes palimpsest (see Supplement. Archimedes' *Method*, Part 1 for the history of the palimpsest), the end of *Measurement of a Circle* is followed by the *Stomachion*. The name comes from the idea that the problem (or puzzle) is so hard to solve that it gives one a bellyache (Eves says the name of this work is *Loculus Archimedius*; see his page 169). There is only one bifolio of the palimpsest (and so only two pages of the original work) which contains material from the *Stomachion*. Heiberg translated the first paragraph of this which, in part, reads (from Netz and Noel's *The Archimedes Codex*, 2007, page 239):

> "...to set out in my investigation into which it is divided, by which (number) it is measured; and further also, which are the angles, taken by combinations and added together; all of the above said for the sake of finding out of the fitting together of the arising figures, whether the resulting sides in the figures are on a line or whether they are slightly short of that but so as toe be unnoticed by sight."

Heiberg made no attempt to interpret the meaning of this. In 1899, an Arabic manuscript was found that that gives a brief description of the problem. From this, it is possible to deduce that the problem is to construct a sqaure from 14 pieces. One solution is the following.



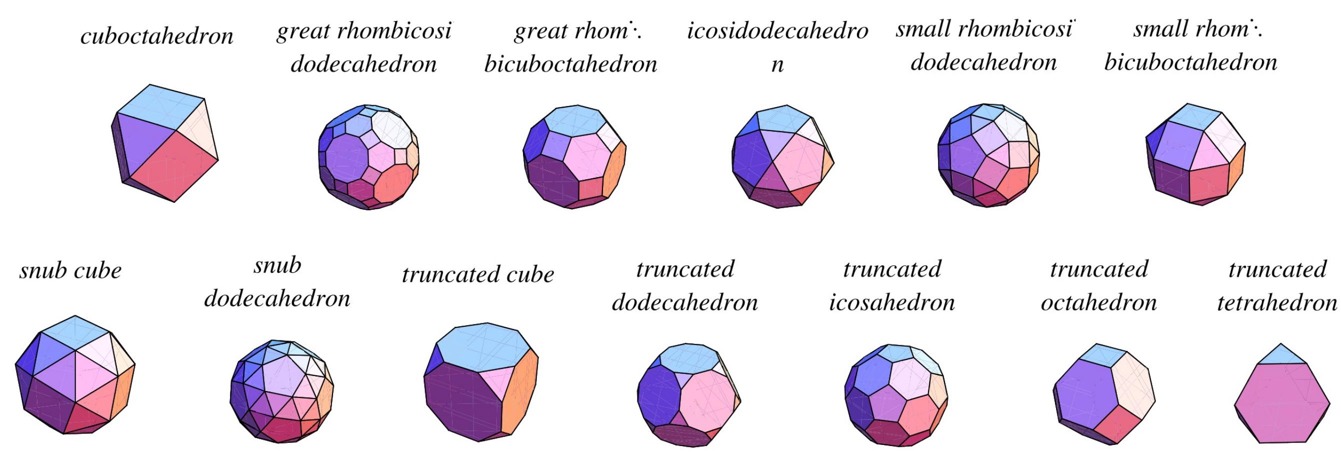
The *Stomachion* puzzle, from MathWorld (accessed 4/28/2024)

While working with the palimpsest in the early 2000s, Wilson and Netz pieced together a second paragraph from the (badly eroded) bifolio containing the *Stomachion* (from *The Archimedes Codex*, page 255):

> "...there is not a small number of figures made of them [the pieces], because of it being possible to rotate them into another place of an equal and equiangular figure, transposed to hold another position..."

This statement by Archimedes involving rotations and transpositions lead Wilson and Netz (and also Fabio Acerbi) to come to the conclusion that the *Stomachion* was about the *number* of ways the pieces could be used to assemble a square. Four mathematicians (Persi Diaconis, Susan Holmes, Ron Graham, and Fan Chung) set out to solve the problem (with the goal, ideally, of solving with techniques that would be known to Archimedes). Through a use of symmetries (so an application of what, today, would lie in the field of group theory) they found 536 "basic solutions," each of which through certain rotations would generate 32 solutions. So they showed that the total number of solutions is $32 \times 536 = 17{,}152$. The problem was also studied by computer scientist Bill Cutler, who wrote an algorithm to go through all potential arrangements of the pieces. He also showed the number of solutions is 17,152. The are of math involved in the counting of such things is known as combinatorics. This material is covered at ETSU in the class Applied Combinatorics and Problem Solving (MATH 3340). See my online notes for Applied Combinatorics and Problem Solving (in preparation) for more details. The results of the study of the *Stomachion* shows that it is "the earliest evidence, anywhere, of the science of combinatorics" (as Netz and Noel put it, page 260). The palimpsest team presented their study of the *Stomachion* in: R. Netz, F. ARceri, and N. Wilson, "Towards a Reconstruction of Archimedes' Stomachion," *SCIAMVS*, **5**, 67–99 (2004). This is available online on the *SCIAMVS*, Sources and Commentaries in the Exact Sciences webpage (accessed 4/28/2028). This paper includes images of the front and back of the bifolio that contains the *Stomachion* material.

Note AW2.J. We know from Pappus' (circa 290 CE–circa 350 CE) *Collection* (or *Synagoge*), Book V, that Archimedes studied "semi-regular polyhedra." Recall that a regular polyhedron (or regular solid, or Platonic solid) is (a) bounded by equal regular polygons, (b) convex, and (c) has the same number of faces at each vertex (see Theorem 44.4 in the Concluding Remark of Section 5.4. Content of the "Elements").



The semiregular polyhedra of Archimedes, from MathWorld (accessed 4/28/2024)

In Book XIII of Euclid's *Elements*, it is shown that the only regular polyhedra are the tetrahedron, cube, octahedron, icosahedron, and dodecahedron. Pappus gives a comparison of the five regular polyhedra. He then discusses the thirteen *semiregular* solids discovered by Archimedes. Such a solid is required to be convex and has faces that are regular polygons, but some faces are not similar polygons. Many of these solids result from truncating a Platonic solid (that is, by slicing off corners of a Platonic solid). The cuboctahedron, icosidodecahedron, truncated cube, truncated dodecahedron, truncated octahedron, truncated icosahedron, and truncated tetrahedron are such solids. The small rhombicosidodecahedron and small rhombicuboctahedron can be obtained by expansion of a Platonic solid. The great rhombicosidodecahedron and great rhombicuboctahedron can be obtained by expansion of one of the previous nine Archimedean solids. The remaining two, the snub cube and the snub dodecahedron, do not result from these processes. A counting argument can be used to show that there are only thirteen semiregular solids. This note is largely based on the from MathWorld webpage on Archimedean Solids (accessed 4/28/2024), which gives more details and references.

Revised: 4/28/2024