1.10. Arbitrary Bases

Note. In Section 1.7. Positional Numeral Systems we saw that any nonnegative integer can be written as a sum of multiples of nonnegative powers of a given base $b \geq 2$ (see Note 1.7.A). In this section we consider specific examples of bases other than 10 and representation of these numbers using a positional system. We cover addition and multiplication tables and illustrate their use.

Note. Recall that for $b \geq 1$, any nonnegative integer N can be written uniquely in the form

$$N = a_n b^n + a_{n-1} b^{n-1} + \cdots + a_2 b^2 + a_1 b + a_0,$$

where $0 \leq a_i \leq b-1$ for each $i \in \{0, 1, \ldots, n\}$. This is proved in Elementary Number Theory (MATH 3120); see my online notes for that class on Section 13. Numbers in Other Bases (see Theorem 13.3). We then represent N with respect to base b in a positional numeral system as the sequence of basic symbols: $a_n\, a_{n-1} \cdots a_2\, a_1\, a_0$. In this section, as is common whenever considering a setting where more than one base may be used, we represent this representation as $(a_n\, a_{n-1} \cdots a_2\, a_1\, a_0)_b$. When we do not write the base b as a subscript, it should be understood that we are considering the standard base 10.

Note. As an example, suppose we consider base $b = 12$. This results in a *duodecimal* system. This is the topic of Section 14. Duodecimals in Elementary Number Theory (MATH 3120). Since we need symbols for each integer between 0 and

$b - 1 = 11$, we use the usual numerals for 0 through 9 and add the symbols t and e to represent 10 and 11, respectively. We then have:

$$6647 = 3(12^3) + 10(12^2) + 1(12) + 11 = (3t1e)_{12}.$$

To find such a representation, we need a technique for finding the coefficients of the powers of the base.

Note. Let N be a nonnegative integer and let $b \geq 2$. Then we know

$$N = a_n b^n + a_{n-1} b^{n-1} + \cdots + a_2 b^2 + a_1 b + a_0,$$

for unique $0 \leq a_i \leq b - 1$. The proof of this is based on iterated use of the Division Algorithm (as shown in Elementary Number Theory [MATH 3120]). We now illustrate how to use this idea to find the coefficients a_i. If we divide N by b then we have

$$N/b = a_n b^{n-1} + a_{n-1} b^{n-2} + \cdots + a_2 b + a_1 + a_0/b = N' + a_0/b.$$

That is, N divided by b is N' (N' is the "quotient") with remainder a_0 (this is the Division Algorithm; see Theorem 1.2 in my online notes for Elementary Number Theory on Section 1. Integers). Applying the Division Algorithm to N' we next have:

$$N'/b = a_n b^{n-2} + a_{n-1} b^{n-3} + \cdots + a_2 + a_1/b = N'' + a_1/b.$$

That is, N' divided by b is N'' (the quotient) with remainder a_1. Hence, by iterating the Division algorithm and applying it to the quotient of the previous application gives the desired values of the a_i as remainders.

Note. Eves illustrates this idea on page 26 by expressing 198 in base 4, and expressing 6647 in base 12. We have 198/4 has quotient 49 with remainder 2, so $a_0 = 2$; 49/4 has quotient 12 with remainder 1, so $a_1 = 1$; 12/4 has quotient 3 with remainder 0, so $a_2 = 0$; and 3/4 has quotient 0 (so the iteration ends with this step) with remainder 3, so $a_3 = 3$. Since the process has stopped, we have $(a_3\, a_2\, a_1\, a_0)_4 = (3012)_4 = 198$. Similarly in duodecimals for $N = 6647$, we have 6647/12 has quotient 553 with remainder $11 = e$, so $a_0 = e$; 553/12 has quotient 46 with remainder 1, so $a_1 = 1$; 46/12 has quotient 3 with remainder $10 = t$, so $a_2 = t$; 3/12 has quotient 0 with remainder 3, so $a_3 = 3$. The process stopped since the last quotient was 0, and we have $(a_3\, a_2\, a_1\, a_0)_{12} = (3t1e)_{12} = 6647$ (as we saw above).

Note. When computing sums and products (and differences and quotients) in a positional numeral system (or "place-value system"), we need only know the sum and products of basic symbols $0, 1, \ldots, b-1$. That is, we need to know our addition and multiplication tables. For base 4 we have:

Addition

	0	1	2	3
0	0	1	2	3
1	1	2	3	10
2	2	3	19	11
3	2	10	11	12

Multiplication

	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	10	12
3	0	3	12	21

Notice through symmetry (and the obvious entries which involve the additive identity 0 and the multiplicative identity 1) how few entries need to be memorized. Consider the sum and product of $(3012)_4$ and $(233)_4$. In the usual hand-computation style we have (eliminating the base 4 as a subscript):

$$\begin{array}{r} 3012 \\ +233 \\ \hline 3311 \end{array} \qquad \begin{array}{r} 3012 \\ \times 233 \\ \hline 21102 \\ 21102 \\ 12030 \\ \hline 2101122 \end{array}$$

We can similarly perform division using the multiplication table. In Elementary Number Theory (MATH 3120), the base 12 multiplication table is given in Section 14. Duodecimals. Its use in multiplication and division is illustrated in those online notes.

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