1.7. Positional Numeral Systems

Note. In this section we define a "ciphered numeral system" and illustrate it with the system example used by the Greeks as far back as 450 BCE.

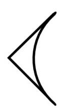
Definition. A *positional numeral system* has a base $b > 1$ and a set of basic symbols for $0, 1, 2, \ldots, b-1$. The b basic symbols are called the *digits* of the system.

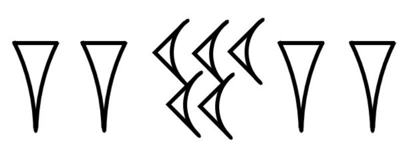
Note 1.7.A. Any nonnegative integer N can be written uniquely in the form

$$N = a_n b^n + a_{n-1} b^{n-1} + \cdots + a_2 b^2 + a_1 b + a_0,$$

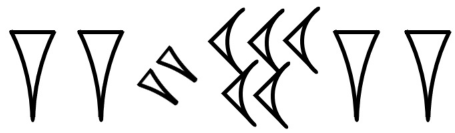
where $0 \le a_i \le b-1$ for each $i \in \{0, 1, \ldots, n\}$. This is proved in Elementary Number Theory (MATH 3120); see my online notes for that class on Section 13. Numbers in Other Bases. The proof is based on repeated application of the Division Algorithm (see Theorem 1.2 of Section 1. Integers of the Elementary Number Theory notes). We then represent N with respect to base b as the sequence of basic symbols: $a_n \, a_{n-1} \cdots a_2 \, a_1 \, a_0$. Notice that the position of the basic symbol in this sequence determines the power of the base by which it is multiplied. For example, in our base 10 Hindu-Arabic numeral system the number 247 stands for $2 \times 10^2 + 4 \times 10^1 + 7$. Base 2, this number is 1111011 because $247 = 1 \times 2^7 + 1 \times 2^6 + 1 \times 2^5 + 1 \times 2^4 + 0 \times 2^3 + 1 \times 2^2 + 1 \times 2 + 1$. Of course, for unambiguity, a positional numeral system requires a symbol for 0.

Note. We introduced the Babylonian numerals used for 1 and 10 in Section 1.4. Simple Grouping Systems and explained how numbers 1 through 59 are represented. Using a positional numeral system, sometime between 3000 BCE and 2000 BCE developed a representation of numbers using a mixed system based on base 60 (i.e., sexigesimal representation). It is mixed because of the fact that there is still a separate symbol for 10. If we think in terms of counting Babylonian style, we might produce a sequence of 1's:

Then when we reach 10, we replace the 10 symbols for 1 with a single symbol for 10:  This idea is then extended. When the number 59 is reached (with the technique of Section 1.4. Simple Grouping Systems), then and increase by 1 to produce 60, the rules of a positional system are imposed and a single 1 in the "second position" is introduced. Peter Rudman refers to this as "a sequence of alternating 1-for-10 and 1-for-6 replacements" (see page 89 of his *How Mathematics Happened*). Initially the Babylonian system lacked a zero symbol, leading to ambiguity in reading the Babylonian positional system. For example, the following is a 2 followed by a 52:

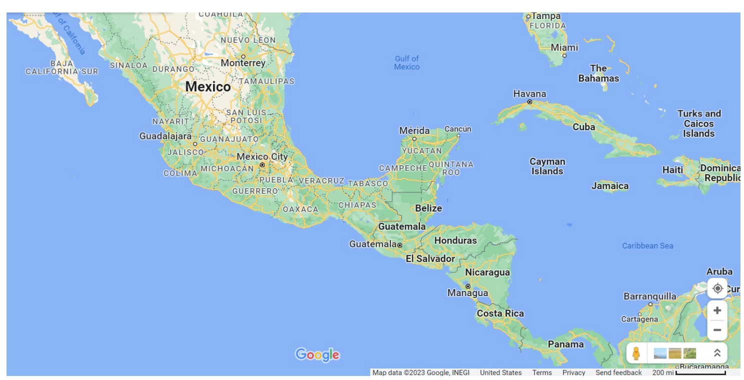


One sexigesimal interpretation is that this represents $2(60) + 52 = 172$. However, another interpretation is that this represents $2(60)^2 + 52 = 3652$. The interpretation is dependent on the setting, and the reader would have to be aware of this setting in order to interpret the written sexigesimal number. A zero symbol was eventually introduced ("around 500 BCE" according to Rudman, and "after 300 BCE" according to Eves). This represents an invention of zero (though it was not recognized as a number, but only as a place holder in a positional numeral system). The zero/place holder was two small, slanted wedges: . We can now represent $2(60)^2 + 0(60) + 52 = 3652$ as:



Before we credit the Babylonians with the invention of the *number* zero, we observe that even as a place holder, their use was inconsistent. They used  to indicate a missing power of 60 *within* a number, and not for any missing power 60 occurring at the *end* of the number. So the previous figure could also represent $2(60)^3 + 0(60)^2 + 52(60) + 0 = 435{,}120$. For this reason, Eves refers to the symbol as "only a partial *zero*" (page 20).

Note. The Maya civilization occupied areas including modern day southeastern Mexico, Guatemala, Belize, and parts of Honduras and El Salvador. All of the Yucatán Peninsula was included in its area.



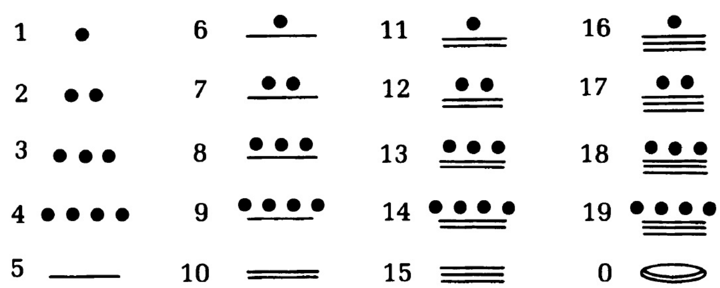
The map above left (and some of the historical information on the Maya) is from the Wikipedia page on May Civilization. The (modern) map on the right is from

GoogleMaps (accessed 5/21/2023). The Preclassic period (circa 2000 BCE–250 CE) sees the beginning of settlements and agriculture in the area. Cities began to appear near the end of this period (including Tikal, El Mirador, Chocol'a, and Komchen). Much of the society collapsed in the first century CE and many of the great Maya cities were abandoned. The Class period (250 CE–900 CE) includes a "Maya renaissance." Construction of cities and monuments (including inscriptions) spread. The largest cities had estimated populations of 50,000 to 120,000; the cities include Tikal (again), Calakmul, Copán, and Quiriguá. During the 9th century, collapse again occurred, though it was a bit more regional this time. Possible explanations of this collapse are warfare, environmental degradation due to overpopulation, and draught. The Postclassic period (950–1539) saw migration to the coasts, and some slow resettling of the abandoned lands. Spanish "explorers" captured the Aztec capital of Tenochtitlan in Mexico in 1521. Assaults on the Maya began in earnest 1523. By 1546, most of the may kingdoms had been conquered and in 1697 the last Maya city fell. As Rudman states in *How Mathematics Happened* on his page 115:

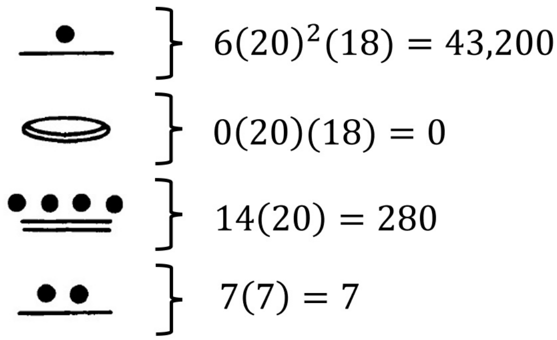
> "Although in a state of decline by the time Spanish conquerors arrived in the sixteenth century, many Mayan texts still existed that possibly could have been able to give a detailed description of Mayan mathematics. Unfortunately, in their zeal to eradicate the Mayan religion and convert the Maya to Christianity, the Spaniards burned most of the Mayan texts. Only three major texts survived: the Dresden Codex, the Paris Codex, and the Madrid Codex, so named for the cities where they are now preserved. Only one of these, the Dresden Codex that records

astronomy observations, is the source of most of what we know about Mayan mathematics. Probably written around the year 1100, this codex contains seventy-eight pages made from wood-bark paper, and each page is about 8.5 cm $\times$ 20.5 cm."

Note. The Maya used an essentially base 20 (i.e., vigesimal, though this is not strictly true as we soon explain) positional numeral system. Rudman speculates that this evolved from using fingers and toes to count (see his page 116). They also invented a full-fledged zero symbol. The Maya numerals (as given by Eves) are:



Numbers were written vertically and read from top to bottom. They deviate from a straight vigesimal system in that (according to Eves; see his page 20) for powers of 20 greater than 1 (that is, numbers involving 20^n where $n > 1$) are computed by replacing one of the factors of 20 by 18. So the nth basic symbol (counting from bottom to top in the Mayan presentation) is taken as a multiple of $(18)(20^{n-2})$ when $n \geq 3$ (the first basic symbol [when $n = 1$] is the "ones" and the second basic symbol [when $n = 2$] is multiplied by 20). Eves states that the priest class used this mixed-base system, but that there are reports that a pure vigesimal system was used by the lay class (though no examples survive in written form); see Eves' page 21. In the mixed base system, the number 43,487 is written as:



Notice that, since the Maya system is base 20, its representation requires fewer basic symbols than our base 10 representation! This strange 18-20 mixed-base is related to the fact that the Maya calendar was based on a $360 = (18)(20)$ day year (as Eves states on page 20) consisting of 18 months of 20 days each (plus 5 "evil days"; they were aware of the more accurate 365 day length of a year, though they still counted in terms of a 360 day year). Rudman explores this in more detail and explains it in terms of astronomical observations (see his pages 124 to 126), which he stated above is the content of the Dresden Codex. Rudman refers to the following page of the codex (he gives a version of this in his Figure 3.3.1):

This image is from the Wikipedia page for the Dresden Codex (accessed 5/22/2023). The circled material includes the numbers (read from RIGHT TO LEFT) 2,920 = 5×584, 5,840 = 10×584, = 15×584, and 11,680 = 20×584 (actually, the left-most number is 11,620, but Rudman declares this a copying error). The amount of time it takes the planet Venus to appear in the same position relative to the sun by an observer on Earth (called the *synodic period*) is 584 days. This is a list of observations of the planet Venus and its movement in the night sky. This time would preferably be recorded in years and days. Thus, we have the use of $(20)(18) = 360$ when dealing with the third and later basic symbols. Notice that the number of years can quickly be determined from the circled information by ignoring the bottom two numbers (which determine the multiples of 1 and 20). For example, the left-most number (read from top to bottom) consists of the basic symbols 1, 12, 5, and 0. So this represents $1(20) + 12 = 32$ years (plus the number of days given by the bottom two symbols; $5(20) = 100$ as given in the photo, though it *should be* $8(20) = 160$ to produce the proper multiple of 584 days).

Revised: 6/4/2023