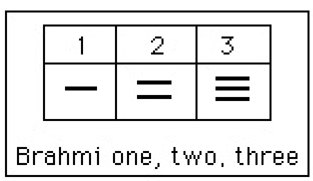
1.9. The Hindu-Arabic Numeral System

Note. Eves only gives one full page of text for this section of the textbook. We largely rely on Chapters 24, 25, and 26 of Georges Ifrah, *The Universal History of Numbers: From Prehistory to the Invention of the Computer* (John Wiley & Sons, 2000) for this section of notes, through we also rely on the (reputable) MacTutor History of Mathematics webpages. The first sentence of Eves summarizes the story: "The Hindu-Arabic numeral system is named after the Hindus, who may have invented it, and after the Arabs, who transmitted it to western Europe." By "Hindu" is meant those who lived in India. We will give a pretty thorough genealogy of the numerals 1, 2, 3, 4, 5, 6, 7, 8, 9, and 0 and show that Eves' statement "who may have invented it" can be replaced with "who invented it!"

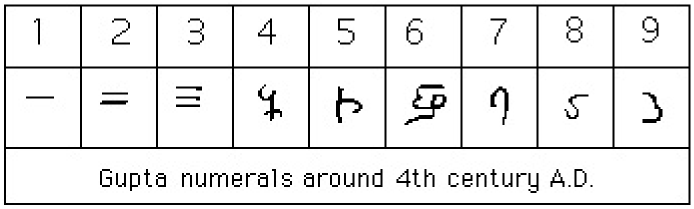
Note 1.9.A. The symbols we are familiar with, "0, 1, 2, 3, 4, 5, 6, 7, 8, 9," take on a physical form close to that we use in the 15th century in Europe. With the invention of the printing press by Johannes Gutenberg (circa 1400-February 3, 1468) around 1450 the symbols became fairly standardized. A major source of our knowledge of the history of the Indian numerals is Islamic mathematician Al-Biruni (September 15, 973–December 13, 1048), who visited India in the 1020s. He wrote 27 works on Indian mathematics and science. We'll explore the spread of the Indian numerals through the Arabic world below. First, we concentrate on the early history and evolution of the symbols. Some of this history is based on the MacTutor History of Mathematics webpage on the Indian Numerals (accessed 5/29/2023).

Note 1.9.B. We begin the story of the Indian numerals with the Brâhmî numerals, which can be traced to the third century BCE. This date is based on the fact that the symbols appear engraved on rocks, temples, and sandstone columns during the time of emperor Asoka (or "Ashoka," who ruled part of the Indian subcontinent between circa 268 and 232 BCE; see Ifrah page 377). (The Brâhmî script was used from around the third century BCE to the 5th century CE in India. During this time Sanskrit, the language of Hinduism and classical Hindu philosophy, was written using the Brâhmî script.) Before this point in history, it is unclear what what motivated the symbols (in particular, for numbers 4, 5, 6, 7, 8, 9). Some ideas (as mentioned by Ifrah) for ancestors of the Brâhmî numerals include: they came from Aramaean numerals (Aram was located in present-day Syria and was part of Babylonia in the 11th-8th centuries BCE), from Karoshthi alphabet (used in the north-western parts of India in present-day Pakistan and Afghanistan), from Brâhmî alphabet, from an earlier alphabetic numeral system, and they came from Egypt. The Brâhmî numerals one, two, and three were:



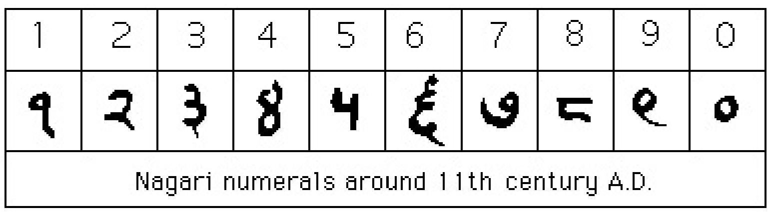
Brahmi one, two, three

There were also Brâhmî symbols for 4, 5, 6, 7, 8, 9, and, as with other non-positional (that is, non place-value) number systems (such as the Greek alphabetic numerals, see Section 1.6. Ciphered Numeral Systems), there were also symbols for 10, 100, 1000, . . . , as well as 20, 30, 40, . . . , 90, and 200, 300, 400, . . . , 900. The Gupta period ranges from the early 4th century CE to the late 6th century CE and marks the time during which the Gupta dynasty ruled parts of India (namely, the northeastern parts). The Gupta symbols are descendants of the Brâhmî numerals (as are other numeral systems which did not survive) and spread as the Gupta empire conquered other territory.



Gupta numerals around 4th century A.D.

The Gupta numerals evolved into the Nagari numerals beginning around the 7th century and continued to develop from the 11th century onward. It is the Nagari numerals which became part of the Arab approach and which al-Biruni wrote about.



Nagari numerals around 11th century A.D.

You'll notice some similarities (and differences) between these numerals and the ones we use. The figures and history of this note are based on the MacTutor History of Mathematics webpage on the Indian Numerals.

Note. We should add a **global warning** about referring to "the Nagari numerals," or "the Gupta numerals," (or "the Attic Greek numeral," etc.). What is given in these notes are *examples* of types of numerals. They would have likely varied from region to region, and certainly they evolved through time.

Note. Georges Ifrah in *The Universal History of Numbers* has his own theory of the origin of the Brâhmî numerals. He claims that they appearance was "autochthonous," that is they appeared with no outside influence. "In all probability, they were created in India, and were the product of Indian civilisation alone" (Ifrah, page 391). He also gives a paleographial table of the evolution of the symbols 1 to 9 from the Brâhmî, through the Gupta, Nâgarî, and European to the modern (see Ifrah, Figures 24.61 to 24.69).

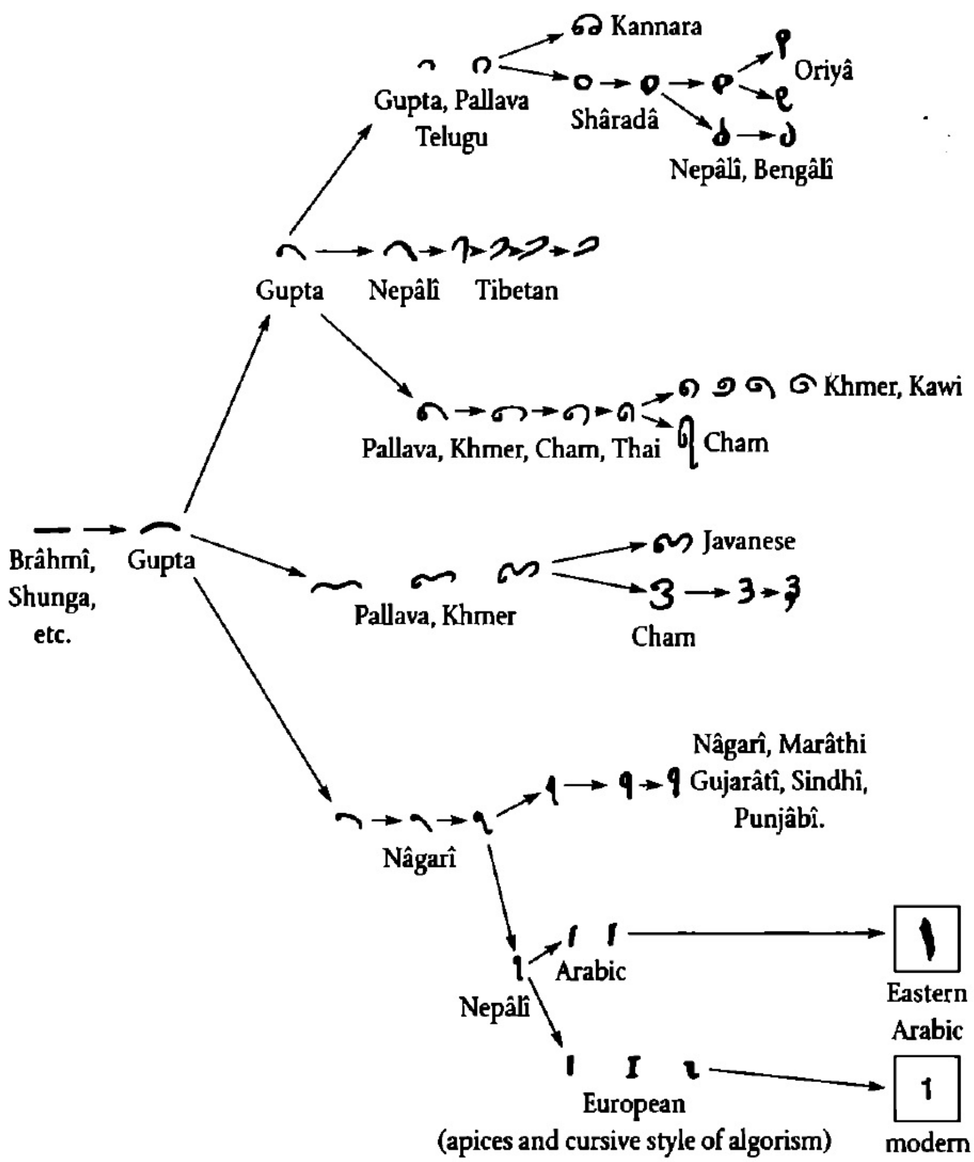


FIG. 24.61. *Origin and evolution of the numeral 1. (For Arabic and European numerals, see Chapters 25 and 26.)*

Note 1.9.C. We now turn to the topics of zero and place-value systems. We saw in Section 1.7. Positional Numeral Systems that these ideas are related. Zero plays the role of a place holder to indicate the absence of certain quantities (such as multiples of 10 or 100 in a base 10 system) in a positional numeral system, but ultimately it realizes its "numberhood" and interacts with other numbers through the arithmetic operations. Ifrah argues (when writing the original French version of his book in 1994) that the oldest known use of zero and of the Indian decimal place-value system appears in the *Lokavibhâga*, dated 458 CE. The *Lokavibhâga*, or "The Parts of the Universe," is the Jain cosmological text: the oldest known Indian text to use zero and the place-value system with word-symbols. (page 419) The date of 458 CE is based on a printed edition of the *Lokavibhâga* which claims that the work was written on a certain day for which astronomical details are given. This date translates to August 25, 458 CE. Mentioning documents written earlier than the middle of the Gupta era (namely, around 450 CE), Ifrah concludes that this is the date of "*the discovery of zero and the place-value system*" (his emphasis; see Ifrah page 420). However, the surviving text of *Lokavibhâga* is a translation written "some considerable time" after the claimed date.

Note 1.9.D. As a side comment, we mention the Bakhshali manuscript. This is a work of mathematics written on birch barks and discovered in 1881 in the village of Bakhshali, Marden (in present-day Pakistan). It includes examples and rules for problems in arithmetic, algebra, and geometry. Of relevance here, is the fact that it includes a place-value system, with zero represented by a dot. The manuscript has received some recent publicity based on carbon dating parts of it in 2017. Three different folios ("pages") produced contradictory dates of 224–383, 680–779, and 885–993 CE. If the first date is correct, this would make it the earliest known use of zero in this role. See the Wikipedia page (the source of the image below) and the MacTutor page on the Bakhshali manuscript. One relevant article on the carbon dating is from *NewScientist* in 2017 at History of Zero Pushed Back 500 Years by Ancient Indian Text (websites accessed 5/30/2023).

The zero symbol in the Bakhshali manuscript.

Note 1.9.E. To further solidify the date of circa 450 CE as the date of introduction of the number zero and a place-value system. Ifrah observes that these ideas "begin to appear frequently in documents from India and Southeast Asia" (Ifrah, page 420). As examples, he mentions astronomer Varâhamihira's it Pañchasiddhântika circa 500 CE and Bhâskhara's commentary on the *Âryabhatîya* in 629 (Ifrah, page 439). But the most complete understanding of zero the number is given by Brahmagupta in 628 in his *Brâhmasphutasiddhânta*. He defined zero as the result of subtracting a number from itself and observed that adding or subtracting zero leaves a number unchanged, and that multiplying by zero yields zero (though he struggled for a complete understanding of zero and division; he stated that "$0/0 = 0$," for example). He described other arithmetic properties of zero, positive, and negative numbers (he called positive numbers "fortunes' and negative numbers 'debts') as follows (Ifrah, page 439):

A debt minus zero is a debt.

A fortune minus zero is a fortune.

Zero (*shûnya*) minus zero is nothing (*kha*).

A debt subtracted from zero is a fortune.

So a fortune subtracted from zero is a debt.

The product of zero multiplied by a debt or a fortune is zero.

The product of zero multiplied by itself is nothing.

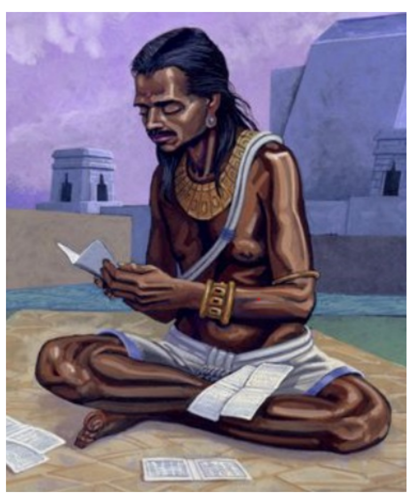
The product or the quotient of two fortunes is one fortune.

The product or the quotient of two debts is one debt.

The product or the quotient of a debt multiplied by a fortune is a debt.

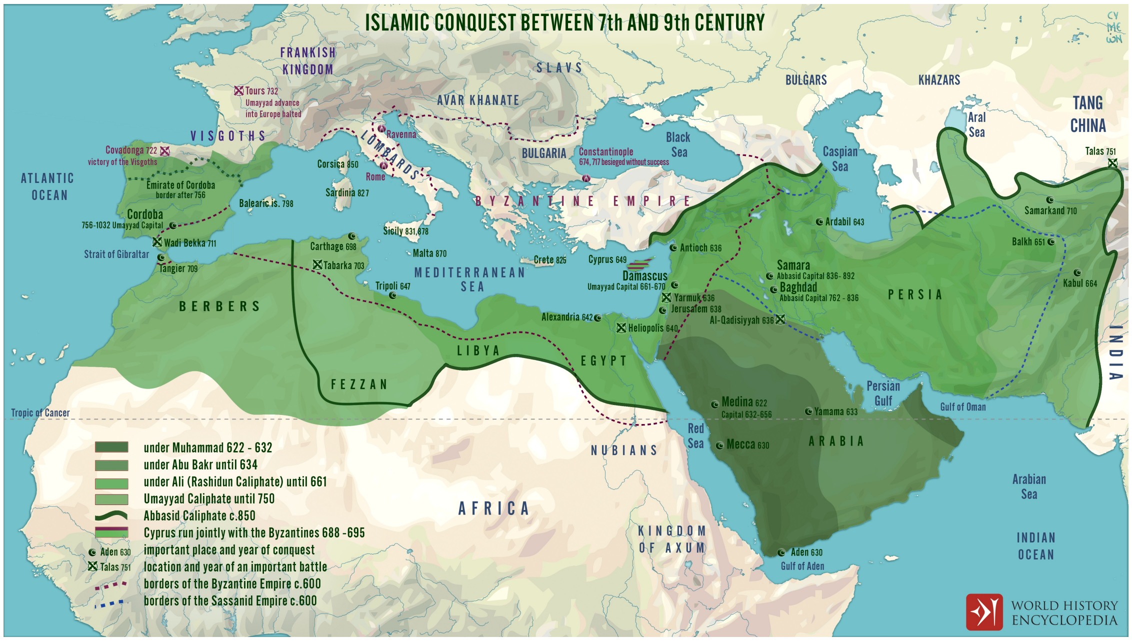
The product or the quotient of a fortune multiplied by a debt is a debt.

Brahmagupta has explicitly stated the "rule of signs" (a negative times a negative is positive, etc.)! *Here* we see a clear foundation of the rules of arithmetic for the first time (in 628). We'll see more of Brahmagupta and the *Brâhmasphutasiddhânta* in Section 7.5. India: General Survey, Section 7.7. India: Arithmetic and Algebra, and Section 7.8. India: Geometry and Trigonometry.



An imagined likeness of Brahmapgupta from Cuemath.com (accessed 5/30/2023)

Note 1.9.F. Now we turn our attention from the Indian/Hindu numerals to their spread through the Arabic world.



From World History Encyclopedia website (accessed 5/31/2023)

Baghdad (the capital of present-day Iraq) became the center of trade and intellectual activity of the Near East in the 9th century. This is where Arabic science and mathematics begins. Between the 8th and 13th centuries, the Arab scholars developed mathematics, astronomy, philosophy, medicine, pharmacy, zoology, botany, chemistry, mineralogy and mechanics. They translated and collected works, including math and science of classical Greece, and universities and libraries sprang up throughout the Arabic world (Ifrah, pages 512 and 513). Following the destruction of the library of Alexandria (first by Christians in the 4th century CE and secondly by the Muslims in the 7th century; see Section 5.1. Alexandria), is when works of classical Greece were translated (of course, we'll never know what did not survive). It is thanks to the work of these Arabic translators that we know as much as we do about the Ancient Greeks today. With the classical results as a foundation and the newer results of the Indians and Chinese, the mathematicians of the Arabic countries were able to develop efficient algorithms and generalizations of problems considered by the Indians and Chinese. The Arabs combined the strict systemization of Greek mathematics and philosophy with the practicality of Indian science. The Arabic scientists questioned some of the doctrines of the ancients and, through synthesis of ideas and experimentation, extended and corrected some of the classical theories (Ifrah, pages 515 and 516).

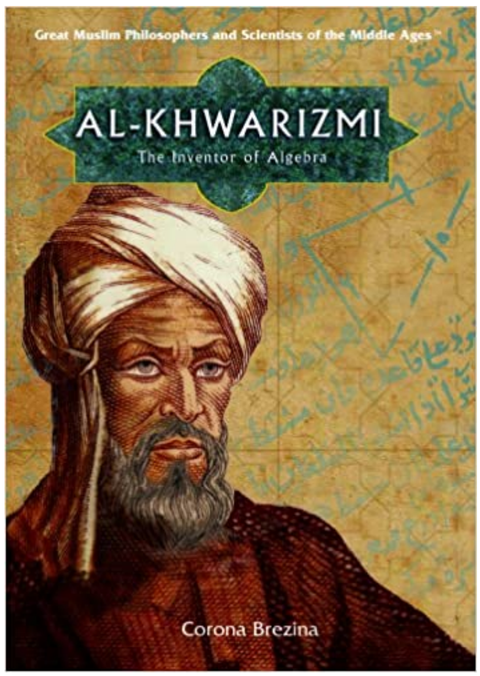
Note 1.9.G. One of the most important sources of information on Indian numerals comes from the mathematician, astronomer, physician, and geographer Abu al-Biruni (September 15, 973–December 13, 1048). During the 1020s he made several visits to India, where he learned Sanskrit and studied Indian science. Al-Biruni wrote his massive *India* in which he described Indian religion, philosophy, marriage customs, the caste system, geography, astronomy, astrology, calendar, writing, and numerals. He also made a new calculation of trigonometric tables based on work of Archimedes (and equivalent to Ptolemy's approach). (This information and the image below of al-Biruni is from the MacTutor pages on Indian Numerals and al-Biruni; accessed 6/1/2023.)

Note 1.9.H. The Arabic historian and sociologist Ibn Khaldun (May 27, 1332–March 17, 1406) in his work *Muqakkimah* (or *Prolegomena* or "Introduction") explains the arrival of Indian numerals in the Arabic world as a result of group of Indian scholars visiting the court of caliph al-Mansur (circa 714–October 6, 775) in 773. These scholars brought information on their science, numerals, and techniques of calculation. (Ifrah, page 529; Ifrah states that the visit was in 776, but this seems to be an error and contradicts the date of 773 which he gives on his page 530.) When learning Indian astronomy, the Arabs had to deal with Indian numerals and calculation with them. Ifrah (on his page 530) quotes A. P. Youschkevich, *Les Mathématique arabes* ("Arabic Mathematics") (Paris: Vrin, 1976) as stating:

> "Three astronomers who worked during the reign of Caliph al-Mansur are known to us, thanks to al-Qifti [an Egyptian Arab historian living from circa 1172 to 1248]: Abu Ishaq Ibrahim al-Fazzari (died c. 777) who first made Arabic astrolabes, his son Muhammad (died c. 800), and finally Ya-qub ibn Tariq (died c. 796), who wrote works dealing with spherical geometry and who also compiled various tables."

Ifrah concludes that it is "quite likely that not only Indian astronomy, but mathematics too, were introduced to the Muslims through the work of Brahmagupta" (page 530), name the *Brahmasphutasiddhanta* ("The Opening of the Universe"). The Wikipedia page on Arabic numerals is less committal and observes that "all Indian texts after Aryabhata I's *Aryabhatiya* used the Indian number system of the nine signs, certainly from this time the Arabs had a translation into Arabic of a text written in the Indian number system." Though the details may differ, the consensus seems to be that the Arab scholars learned Indian astronomy and, in the process, had to learn to deal with the Indian numerals and computations.

Note 1.9.I. We now turn to the Arabic world's most famous mathematician, Abu Ja'far Muhammad ibn Musa Al-Khwarizmi (circa 780–circa 850), who we will see again in Chapter 7, "Chinese, Hindu, and Arabian Mathematics." He was a member of a group of mathematicians and astronomers who worked at the 'House of Wisdom' (*Bayt al-Hikma*), Baghdad's scientific academy. He is known for two works which spread the use of the Indian numerals and methods of calculation with them through the Arab world and, eventually, through the "west."



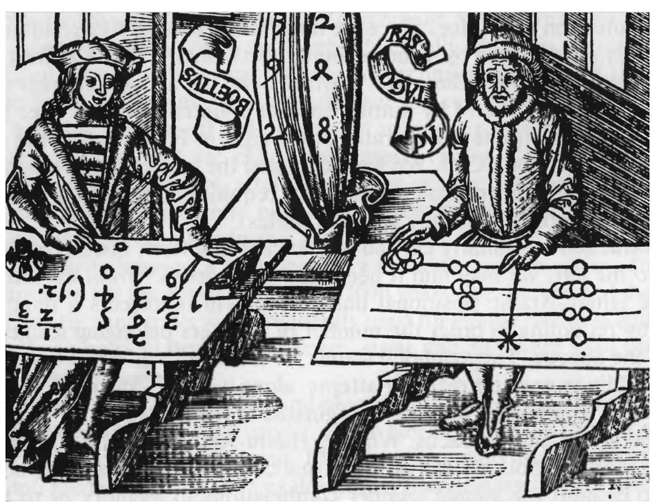
A book describing the life and work of al-Khwarizmi for ages 10 and up; image from Amazon.com (accessed 6/1/2019)

One of the works *Al jabr wa'l muqabala* (*Transposition and Reduction*), deals with the basics of algebra. Copies in Arabic and a medieval Latin translation survived. It is the title of this book that gives us the term "algebra." The technique *al jabr* is the transposing of terms in an equation to get both sides positive (there is still a nebulous understanding of negative numbers). The technique *al-muqabala* is the reduction of similar terms in an equation (i.e., collecting together common terms and simplifying). The other work was *Kitab al jami' wa'l tafriq be hisab al hing* (*Indian Technique of Addition and Subtraction*). An Arabic version did not survive, but some 12th century Latin versions did. It is the first known Arabic book which explains in detail the Indian place-value system, along with examples. al Khwarizmi's work was so influential, that the Latinized version of his name evolved into our term "algorithm" (Ifrah, page 531).

Note. The use of Indian numerals developed separately in the western part (consisting of North Africa and Spain) and eastern part (east of Egypt) of the Arabic world. The version of the numbers that worked its way into Europe came through the western part. The Indian numerals of the western part of the Arabic world were called "ghubar numerals." This is because computations were often done by spreading flour over a board and then tracing the figures; "ghubar" means "dust." The two oldest known documents which refer to Ghubar numerals and calculation date to 874 and 888, so it is likely that the Indian numerals (in the ghubar form) reached North Africa and Spain during the ninth century. From there, they migrated throughout western Europe (Ifrah, pages 536, 537, and 539).

Note. We might expect that once the "Hindu-Arabic numerals" reached Europe, they quickly spread and displaced the other versions. However, Europeans were attached to the "old ways" and were reluctant to accept the new ideas. In fact, centuries passed before the new numerals became dominant (Ifrah, page 577). There was also reluctance in accepting the new numerals when they were introduced to the Arabic world, which also remained attached to their ancestral methods of counting an calculating (Ifrah, page 541). This is not unusual; simply consider the 20th century attempts to adopt the decimal system in the United States.

Note. Two common ways of performing summations involved (1) the use of counters on a ruled table (or 'abacus'), and (2) using written Arabic numerals (users became known as "algorists"). The first technique is a descendant of the Greek and Roman traditions. The second technique was often attributed to the Arabs, but is in fact due to the Indians. The use of the Roman abacus "constituted an obscure and complex art, the specialist preserve of a privileged caste, whose members had been through a long and rigorous training which had allowed them to master the mysterious and infinitely complicated use of the classical (Roman) counter-abacus." It could take years to to master multiplication and division on the abacus (Ifrah, page 577)!



A close-up of "The Abacist versus the Algorist" from Gregor Reisch, *Margarita Philosophica*, Strassbourg 1504 (from Eves' page 24).

The algorist, though, must only memorize a multiplication table involving the 10 numerals in order to perform more complicated multiplication (and, its sibling, division). However, some in the Church "put it about that arithmetic in the Arabic manner, precisely because it was so easy and ingenious, reeked of magic and of the diabolical: it must have come form Satin himself. It was only a short step from there to sending over-keen algorists to the stake, along with witches and heretics. And many did indeed suffer that fate at the hands of the Inquisition" (Ifrah, pages 588 and 589). This religious domination of knowledge slowed the spread of new ideas (mathematical and otherwise) for several centuries (Ifrah, page 588). In addition, western Europe was repeatedly the victim of epidemics, famine, warfare, and political instability for much of this time. "The so-called 'Carolingian renaissance' in the Benedictine monasteries of the ninth century may have revitalised the idea and structure of education in the era of Charlemagne [April 2, 747–January 28, 814] and also laid the bases of mediaeval philosophy, but it actually brought only minor and temporary relief to the general situation" (Ifrah, page 578).

Note. Gerbert of Aurillac (circa 946–May 12, 1003), who would become Pope Sylvester II on April 2, 999, was prominent for his work in math and science (as evidenced by the fact that there is a MacTutor webpage on Gerbert of Aurillac, the source of the image of him below; accessed 6/3/2019). He was born in France and spent his childhood there, but visited Spain from 967 to 970 where he learned about Arabic math and science. He taught in a school of the Church in Reims, France and his teachings spread an interest in mathematics. "And it was Gerbert who first introduced so-called Arabic numerals into Europe. Arabic numerals, indeed—but alas, only the first nine! He did not bring back the zero from his Spanish sojourn, nor did he include Indian arithmetical operations in his pack. ... Gerbert's initiative actually met fierce resistance: his Christian fellows clung with conservative fervour to the number-system and arithmetical techniques of the Roman past. The time was simply not ripe for a great revolution of the mind" (Ifrah, page 579). We will see Gerbert again in Section 8.1. "The Dark Ages" (The Middle Ages).

Gerbert of Aurillac, or Pope Sylvester II (circa 946–May 12, 1003)

Note. The first appearance of the Arabic numerals in "the west" is in the *Codex Vigilanus*, an illuminated (i.e., illustrated) collection of historical documents about Spain covering antiquity up to the 10th century and copied in northern Spain in 976. The second appearance is in the *Codex Aemilianensis*, which was copied from *Vigilanus* in the year 992. The versions of the Arabic numerals presented in these are close to the ghubar versions of the western Arabs (mentioned above). As the numerals spread through manuscripts of Europe, the shapes and styles of the numerals varied from place to place and from time to time. However, the spread of the numerals was not so much by written manuscripts, but by word-ofmouth on the use of a computing device called *Gerbert's abacus* (remember that an "abacus" is a table on which counters representing numbers are moved around as computations are done, not a system of wires along which beads slide [this is a "Chinese abacus"]). Gerbert's abacus involved removing multiple unit counters and replacing them with single labeled counters in each decimal column. So, for example, if four unit counters were present in a column, then they were replaced by a single counter (called an *apex*) with a numeral in Arabic script on it. It column contained no unit counters then it was left empty (so there was an an apex with a zero label). This resulted in an abacus that was identical in structure to the abacus used with Roman numerals, but was simpler to use. (Ifrah, pages 579, 580, and 582)

Note. During the Crusades (1095 to 1270) "Christian knights and princes tried to impose their religion and traditions on the Infidels of the Middle East" (as Ifrah puts it on page 586). This resulted in, among other things, numerous contacts with the Islamic world, including clerks traveling with the crusaders learning the Arabic numerals and their arithmetical applications. Gerbert's abacus was gradually replaced with the use of the numerals written on sand or dust (instead of engraved on apices)and the columns of the abacus also disappeared. This required the use of zero to eliminate ambiguity. After this long process, the Hindu-Arabic numbers had spread through Europe. (Ifrah, pages 586 and 587)

Note. We mention one additional instrumental character in the story of the spread of the Hindu-Arabic numerals in western Europe. Leonardo of Pisa (or Leonardo Pisano; circa 1170–circa 1250), better known as "Fibonacci," wrote *Liber abacci* (published in 1202, with a second revised edition published in 1228). Ifrah translates *Liber abaci* as "The Book of the Abacus" (see his page 588), though he observes that it has no connection with Gerbert's abacus. In *The Man of Numbers: Fibonacci's Revolution* (Walker & Company, 2011), Keith Devlin states: "*Liber abbaci* translates as 'Book of Calculation.' The intuitive translation 'Book of the abacus' is both incorrect and nonsensical..." (see Devlin's page 11). Amazingly, *Liber abaci* was not translated into English until 2002 (which is 12 years after the 6th edition of Eves' *An Introduction to the History of Mathematics*)! We'll explore Leonardo of Pisa in more detail in Section 8.3. Fibonacci and the Thirteenth Century, and consider *Liber abaci* itself in Supplement. Leonardo of Pisa (Fibonacci) and the *Liber abbaci*. As a concluding remark, we mention that Leonardo of Pisa uses the term "zephirum" for "0" and this gives rise to our modern term "zero" (Ifrah, page 589).

Image from the MacTutor biography of Fibonacci (accessed 6/4/2023)