11.2. Zeno's Paradoxes

Note. Zeno of Elea (490 BCE–425 BCE) is well-known for his "paradoxes" concerning motion, which are based on the concept of infinity. None of his original work exists but is known to us through the writings of Plato (427 BCE–347 BCE) in *Parmenides* and Aristotle (384 BCE–322 BCE) in *Physics*.

From the MacTutor biography webpage on Zeno (accessed 5/2/2024)

Note 11.2.A. In considering the quadrature of the circle (see Section 4.7. Quadrature of the Circle, Antiphon of Rhamnus (480 BCE–411 BCE) proposed inscribing regular polygons in a circle by iteratively doubling the number of sides. "[B]y continuing this process, we should at length exhaust the circle" (Heath's *History, Volume 1*, page 271). We know of this because Aristotle wrote of it in hsi *Physics*, in which he declared Antiphon's idea a fallacy, through Aristotle didn't have the background to offer a mathematical refutation. Also in Heath's *History, 1* (page

272) he describes the response to Zeno's related ideas by the mathematical community:

> "The mathematicians, however, knew better, and, realizing that Zeno's arguments were fatal to infinitesimals, they saw that they could only avoid the difficulties connected with them by once for all banishing the idea of the infinite, even the potentially infinite, altogether from their science. . . [and] contented themselves with finite magnitudes that can be made as great or as small *as we please*. If they used infinitesimals at all, it was only as a tentative means of *discovering* propositions; they *proved* them afterwards by rigorous geometrical methods." [emphasis is Heath's]

These geometrical methods are the method of exhaustion of Eudoxus (408 —sc bce–355 BCE) (see Section 11.3. Eudoxus' Method of Exhaustion) which was very successfully employed by Archimedes (see Section 6.2. Archimedes and the several supplements to that section).

Note 11.2.B. Based on Plato's comments in *Parmenides*, it appears that Zeno only wrote one work (which is no longer surviving) and that Zeno was neither a mathematician nor a physicist (likely, he was a philosopher). He is remembered for his "paradoxes" concerning motion. One of his arguments, "The Dichotomy," claims that no motion can exist because, to move from one place to another, an object must first move half the distance between the places. Before this, the object must move half of *that* distance (i.e., 1/4 of the distance is between the objects), and so forth without end. Of course, an easy resolution to this in modern terms is that it travels through the continuum of position while moving through the continuum of time, so that smaller and smaller increments of time yield smaller and smaller increments of motion; the infinite number of increments and times summing to something finite (in the described case, we have a geometric series with ratio 1/2).

Note 11.2.C. Bertrand Russell (May 18, 1872–February 2, 1970), in his *The Principles of Mathematics* (1903), addresses a modern solution to Zeno's concerns. He states (on pages 347 and 348):

> "After two thousand years of continual refutation, these sophisms [false arguments] were reinstated, and made the foundation of a mathematical renaissance, by a German professor who probably never dreamed of any connexion between himself and Zeno. Weierstrass [October 31, 1815–February 19, 1897], by strictly banishing all infinitesimals, has at last shown that we live in an unchanging world, and that the arrow, at every moment of it slight, is truly at rest. The only point where Zeno probably erred was in inferring (if he did infer) that, because thee is no change, the world must be in the same state at one time as at another."

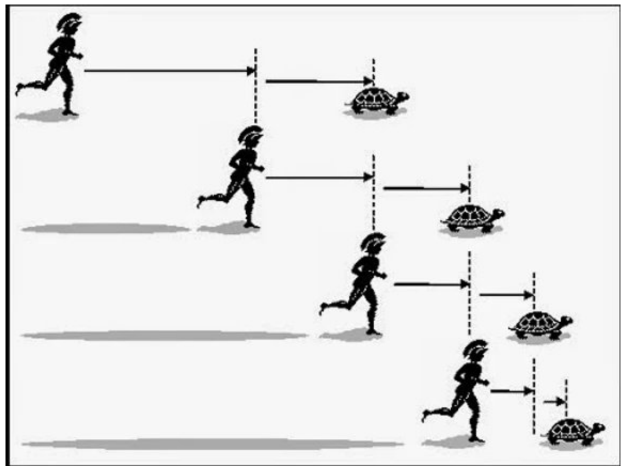
Karl Wilhelm Weierstrass developed a theory of real numbers in which he defined irrational numbers as limits of convergent series. This is similar to an approach taken by Richard Dedekind (October 6, 1831–February 12, 1916) and Georg Cantor (March 3, 1845–January 6, 1918) in developing the real numbers as a complete ordered field in which Cauchy sequences of rational numbers are used to define real numbers. These ideas are developed in Analysis 1 (MATH 4217/5217) and covered in Section 2-3. The Bolzano-Weierstrass Theorem (this section of my online notes includes a brief biography of Weierstrass) and Supplement. The Real Numbers are the Unique Complete Ordered Field. The use of rational numbers and Cauchy sequences thereof allows for the avoidance of a hands-on need for infinitesimals. The presentation given in Analysis 1 is the "standard" modern interpretation of the real number line (and, therefore, of the continuum).

Note 11.2.D. Zeno gives four arguments on the subject of motion. The first two arguments as given by Aristotle in his *Physics* are as follows (quoting from Heath's *History, Volume 1*, pages 275 and 276):

1. The *Dichotomy*. "There is no motion because that which is moved must arrive at the middle)of its course) before it arrives at the end." (And of course it must traverse the half of the half before it reaches the middle, and so on *ad infinitum*.)

2. The *Achilles*. "This asserts that the slower when running will never be overtaken by the quicker; for that which is pursuing must first reach the point from which that which is fleeing started, so that the slower must necessarily always be some distance ahead."

The meaning of the *Dichotomy* is fairly clear, as discussed in Note 11.2.B. The meaning of the *Achilles* is similar to that of the Dichotomy. The idea is that when two runners (commonly taken to be Achilles and a tortoise, thus the name of this "paradox") start out separated by a distance, the faster one (assumed to be behind the slower one) must first go to the initial position of the second runner. But in that time, the second runner has moved through an additional distance to a third position The first runner then must go to this third position, but the second runner has now moved to a fourth position, and so forth. This is illustrated in the image below.



From Quora.com Zeno's Paradox website (accessed 5/2/2024)

Both of these two "paradoxes" are easily explained away with modern elementary mechanics. In the *Dichotomy*, we let the position x of "that which is moved" be given by a function of time or the form $x = vt$ (assuming the position at time $t = 0$ is $x = 0$ and that the moving object as constant velocity v); this is uniform *rectilinear motion*. We might measure v in m/sec, t is sec, and x in m. For the object to move from $x = 0$ to $x = 1$, it must first reach the position $x = 1/2$. This occurs at time $t = 1/(2v)$. Before that, it must reach the position $x = 1/4$, which it does at time $t = 1/(4v)$. In general, it must reach the position $1/2^n$ and it does so at time $t = 1/(2^n v)$. The spatial parameter (or "position") is infinitely divisible, as is the temporal parameter (or "time"). Similarly, in the *Achilles* paradox, we could take the position x_A of the fast runner, Achilles, to be $x_A = v_A t$ and the position x_T of the slow runner, the tortoise, to be $x_T = v_T t + d$ where $v_A > v_T$. Notice that Achilles and the tortoise are initially a distance d apart. As in the

Dichotomy, it Achilles traverses the distance d in a time $t_1 = d/v_A$. But in this time, the tortoise moves to position $v_T t_1 + d = v_T(d/v_A) + d = (v_T/v_A)d + d$; a position now $(v_T/v_A)d$ units ahead of Achilles' position. Next, Achilles covers this distance in $t_2 = ((v_T/v_A)d)/v_A = (v_T/v_A^2)d$. Again, in this time the tortoise moves forward $t_2 v_T = ((v_T/v_A^2)d)v_T = (v_T/v_A)^2 d$, a distance next covered by Achilles in a time of $t_3 = ((v_T/v_A)^2 d)/v_A = (v_T^2/v_A^3)d$. Similarly, at the nth step, Achilles covers the distance between him and the tortoise in a time $t_n = (v_T^{n-1}/v_A^n)d$. We can now sum the series of these times to get the time when Achilles and the tortoise are at the same position:

$$\sum_{i=1}^{\infty} t_n = \sum_{i=1}^{\infty} \frac{v_T^{n-1}}{v_A^n} d = \frac{d}{v_A} \sum_{i=1}^{\infty} \left(\frac{v_T}{v_A}\right)^{n-1}$$

$$= \frac{d}{v_A} \sum_{i=0}^{\infty} \left(\frac{v_T}{v_A}\right)^{n} = \frac{d}{v_A} \left(\frac{1}{1 - (v_T/v_A)}\right) = \frac{d}{v_A - v_T}.$$

Notice that $v_A - v_T$ is the rate at which Achilles closes in on the torus, so this time is what we would expect. This solution is in the spirit of Zeno's paradox, since it addresses the infinite number of steps that so concerns Zeno. A simpler solution, though, can be found by simply setting Achilles position equal to that of the tortoise and then solving for the time:

$$x_A = x_T \text{ implies } v_A t = v_T t + d \text{ or } (v_A - v_T)t = d \text{ or } t = d/(v_A - v_T).$$

Note 11.2.E. Zeno's second two arguments on the subject of motion as given by Aristotle in his *Physics* are as follows (quoting from Heath's *History, Volume 1*, page 276):

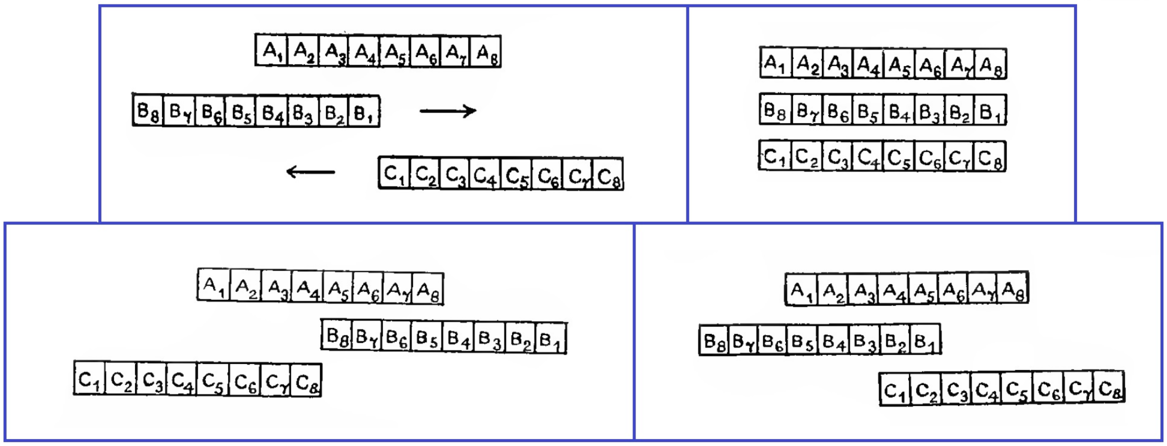
3. The *Arrow*. "If says Zeno everything is either at rest or moving when it occupies a space equal (to itself), while the object moved is always in the instant, the moving arrow is unmoved."

4. The *Stadium*. "The fourth is the argument concerning the two rows of bodies each composed of an equal number of bodies of equal size, which pass one another on a race-course as they proceed with equal velocity in opposite directions, one row starting from the end of the course and the from the middle. This, he thinks, involves the conclusion that half a given time is equal to its double."

The assumptions in these two "paradoxes" are that both space and time are not infinitely divisible and that they are made of up of indivisible elements. For time, the indivisible element is an *instant*. Heath's explanation of the *Arrow* paradox is (*History, Volume 1*, page 276):

> "It is strictly impossible that the arrow can move in the *instant*, supposed indivisible, for, if it changed its position, the instant would be at once divided. Now the moving object is, in the instant, either at rest or in motion; but, as it is not in motion, it is at rest, and as, by hypothesis, time is composed of noting but instants, the moving object is at rest." [Heath's italics]

We take a look at a quantitative evaluation of these ideas below. The story of the *Stadium* is a matter of some debate. The story is illustrated in the figure below. Let the "rows of bodies" be A, B, and C, where row A consist of bodies A_1, A_2, $\dots$, A_8, and similarly for rows B and C (figure, upper left). That is, for the sake of illustration, suppose there are 8 bodies in each row. Now these "bodies" are to be interpreted as indivisible elements of position (or "space"). Initially, take row A to be stationary with row B moving to the right starting with B_1 directly below A_4, and with row C moving to the left starting with C_1 directly below A_5.



From Heath's *History, Volume 1*, pages 277 and 282

There will be some instant when rows B and C will be exactly under row A (figure, upper right). Later there is an instant when the alignments of rows B and C have been interchanged from their initial alignments (figure, lower left). Aristotle explains that Zeno reaches the conclusion that "C_1 is the same time in passing each of the B's as it is in passing each of the A's" (Heath's *History, Volume 1*, page 282). Since row A and row B contain the same number of bodies ("instants"), then the amount of time for C_1 to pass half the A's (from figure upper left to upper right) is the same as the amount of time for C_1 to pass all the B's (and an equivalent amount of time to pass *all* the A's; also from figure upper left to upper right). That is, C_1 passes half the A's in the same amount of time that it passes all of the A's. This contradiction is the paradox, as seen by Aristotle. Aristotle then criticizes Zeno for understanding the difference between absolute and relative motion. V. Brochard, G. Noël, and B. Russell (for references and more details, see

N. Booth, "Zeno's Paradoxes," *The Journal of Hellenic Studies*, **77**(2), 187–201 (1957)) gave an alternative interpretation to the *Stadium*. They argue that Zeno's plan is to show that time is not made up of indivisible elements (i.e., instants). They say to consider the state when the B's have moved one place to the right and the C's have moved one place to the left (figure, lower left). Zeno's hypothesis that time consists of indivisible instants, and B_1 and C_1 are indivisible elements of space implies, since B_1 starts strictly to the left of C_1 and ends strictly to the right of C_1 (figure upper left to lower right), that there must be an instant when B_1 was vertically over C_1. But the motion has taken place in an indivisible instant. So either B_1 and C_1 have not crossed (so that there is no motion) or the instant this motion is divisible after all (in which case it is equal to an instant half its size; thus the wording that there is a given time such that "half a given time is equal to its double").

Note 11.2.F. Zeno four arguments on the subject of motion concern two hypotheses. The first two arguments (Dichotomy and *Achilles*) are based on the hypothesis that continuous magnitudes are divisible *ad infinitum*. The third and fourth arguments (*Arrow* and *Stadium*) are based on the hypothesis that continuous magnitudes are made up of indivisible elements. Heath states (backed up referencing Bertrand Russell; see *History, Volume 1*, page 279) that the objections raised based on the first hypothesis were not formalized until George Cantor (March 3, 1845–January 6, 1918) introduced his theory of continuity and infinity (which is briefly addresses in Analysis 1 [MATH 4217/5217] in Section 1.3. The Completeness Axiom). Heath quotes Bertrand Russell (May 18, 1872–February 2, 1970) from his *Principles of Mathematics*, Cambridge University Press (1903). Russell's book is in the public domain and can be found online on the University of Massachusetts, Amherst website (accessed 5/3/2024); this is not to be confused with his *Principia Mathematica* that he later coauthored with Alfred North Whitehead. My paraphrasing of Russell's comments is the following. He states that motion consists merely of occupation of different places at different times. There is no transition from place to place and no such thing as velocity or acceleration. However, velocity and acceleration do exist in the sense of real numbers which result from taking limits of difference quotients, as velocity and acceleration are defined in Calculus 1 (MATH 1910), Calculus 2 (MATH 1920), and Technical Physics 1—Calculus Based (PHYS 2110). The rejection of velocity and acceleration as physical facts (i.e., a property belonging to a moving object at each instant of time) is "imperative" because of Karl Weierstrass' rigorous handling of calculus given in the late 19th century.

Note 11.2.G. As some final personal observations, I would observe that the ideas of velocity and acceleration are just as "real" as are the ideas of position and time. I think what Russell is pushing is the idea that all of these concepts are results of a *mathematical model* of the physical situation. We define position as a function of time (we we did in Note 11.2.D when discussing the *Achilles* argument), so that velocity and acceleration are then functions based on the position function (namely, the first and second derivatives of position with respect to time). We treat both position and time as represented by real numbers, so that a continuum is then built into the model. The continuum nature of the real line is dealt with in the 19th century by Augustin-Louis Cauchy (August 21, 1789–May 23, 1857), Richard Dedekind (October 6, 1831–February 12, 1916), and Georg Cantor (March

3, 1845–January 6, 1918); see my online notes for Analysis 1 (MATH 4217/5217) on Supplement. The Real Numbers are the Unique Complete Ordered Field. With all this talk of continuity, one has to wonder about the quantum nature of the universe. This relates to a very different mathematical model for physics and one that is only relevant in some very specific settings, namely the behavior of very small things on a very small scale. In *this* model, a particle such as an electron in an orbital of an atom can be observed in that orbital, then observed again later and it may be in another orbital. But the electron will not be observed *between* the orbitals. It makes a "quantum leap" from one orbital to another (by absorbing or emitting a photon). That is, a physical electron does not have a position described by a continuous parameter! However, the model of quantum mechanics involves continuous wave functions which occupy an infinite dimensional vector space which also forms a continuum (the space is called a *Hilbert space*). For some details on quantum mechanical models, see my online notes for Applied Mathematics 1 (MATH 5610) on Section 7.3. Basic Concepts and Postulates of Quantum Mechanics. Notice that these notes are set up much like Euclid's *Elements*, in that there are definitions, postulates, theorems, and proofs! I also have more rigorous notes (though they are currently [summer 2024] incomplete) on this topic for the unofficial class "Hilbert Spaces and Quantum Mechanics" (this would be a graduate-level class requiring Real Analysis 1 [MATH 5210], Real Analysis 2 [MATH 5220], and Fundamentals of Functional Analysis [MATH 5740], at a minimum). Of particular relevance to this conversation is Section I.5. Wave Mechanics of a Single Particle Moving in One Dimension.

Revised: 5/4/2024