11.3. Eudoxus' Method of Exhaustion

Note. We have encountered the method of exhaustion in the work of Euclid (Book XII of the *Elements*; see Note 5.4.P of Section 5.4. Content of the "Elements") and repeatedly in the work of Archimedes (see Supplement. The Content of Archimedes' Work, Part 1 and Supplement. The Content of Archimedes' Work, Part 2). In this section we give more information on Eudoxus, with special attention to his contributions to the method of exhaustion.

Note 11.3.A. Morris Kline in his *Mathematical Thought from Ancient to Modern Times, Volume 1* (Oxford University Press, Volume 1) states (page 48): "The greatest of the classical Greek mathematicians and second only to Archimedes in all antiquity was Eudoxus." Heath, in *History, Volume 1* (page 322), uses Apollodorus (circa 180 BCE–circa 120 BCE) as a source for information on Eudoxus. Eudoxus (408 BCE–355 BCE) at age 23 went to Athens for two months where he attended lectures on philosophy by Plato (circa 428 BCE–circa 348 BCE). He also studied geometry with Archytas (circa 428 BCE–circa 350 BCE) in Italy and studied medicine in Sicily. He spent 16 months in Egypt where he studied astronomy and made observations at an observatory there. His observations were recorded in a book (possibly two books), the *Mirror* and the *Phaenomena* (according to Hipparchus [190 BCE–120 BCE]). However, his contributions to astronomy overlap with his mathematical contributions. He introduced a system of concentric spheres to explain the motions of the sun, moon, and planets against the fixed stars. "It was the first attempt at a purely mathematical theory of astronomy, and, with the great and immortal contributions which he made to geometry, puts him in the very first rank of mathematicians of all time" (Heath's *History, Volume 1*, page 323). He went to Cyzicus in northwestern Anatolia (in the Balikesir Province of modern day Turkey). There he formed a large school, which he took to Athens around 368 BCE (see also Note 4.1.G in Section 4.1. The Period from Thales to Euclid).

From the MacTutor biography webpage on Eudoxus (accessed 4/30/2024)

Note. We first discussed Eudoxus in Section 3.5. Discovery of Irrational Magnitudes in connection to his theory of proportions and resulting contributions to Book V of Euclid's *Elements* (see Note 3.5.E). One of his major contributions is this theory of proportions. We have already addressed this in Note 5.4.J of Section 5.4. Content of the "Elements". There, the Eudoxian definition of proportion is quoted from the *Elements*, Book V, Definition 5, and explained. So we now move on to the method of exhaustion.

Note 11.3.B. In Proclus's commentary on Eudemus' *History of Geometry*, it is suggested that Eudoxus' ideas are a major component of the content of Euclid's *Elements*:

> "Not long after these men [Eudoxus and Theatetus] came Euclid, who brought together the Elements, systematizing many of the theorems of Eudoxus, perfecting many of those of Theatetus, and putting in irrefutable demonstrable form propositions that had been rather loosely established by his predecessors."

See Supplement. Proclus's Commentary on Eudemus *History of Geometry* and notice the summary given of "Proclus' Commentary on the First Book of Euclid's *Elements*, Chapter IV." As commented above, Eudoxus' contributions in terms of the method of exhaustion are prominent in Book XII of the *Elements*. In particular, Euclid uses this method to prove:

Proposition XII.2. Circles are to one another as the squares on their diameters. [That is, the area of a circle is proportional to the square of the diameter, and hence also proportional to the square of the radius. This is how π is defined!]

Archimedes himself attributes the invention of the method of exhaustion to Eudoxus. Several of Archimedes' works start with a letter of introduction to a specific individual. *On the Sphere and Cylinder, Book I* starts with a greeting to Dositheus. He continues (quoting form Thomas Heath's *The Works of Archimedes*, Cambridge University Press, 1897, page 2):

> "...the theorems of Eudoxus on solids which are held to be most irrefragably established, namely, that any pyramid is one third part of the prism which has the same base with the pyramid and equal height, and that any cone is one third part of the cylinder which has the same base with the cone and equal height. For, though these properties also were naturally inherent in the figures all along, yet they were in fact unknown to all the many able geometers who lived before Eudoxus and had not been observed by any one."

Quadrature of the Parabola also starts with a greeting to Dositheus and includes (quoting Heath's *Works*, page 234):

> "...the following lemma is assumed: that the excess by which the greater of (two) unequal areas exceeds the less can, by being added to itself, be made to exceed any given finite area. The earlier geometers have also used this lemma; for it is by the use of this same lemma that they have shown that circles are to one another in the duplicate ratio of their diameters [Euclid's Proposition XII.2 in the *Elements*)...also that every cone is one third part of the cylinder having the same base as the cone and equal height they proved by assuming a certain lemma similar to that aforementioned."

Though not mentioned by name in *Quadrature of the Parabola*, it is clear from his other comments that Archimedes is tying this statement to the one from *On the Sphere and Cylinder, Book I* which does explicitly mention Eudoxus.

Note 11.3.C. We saw in Archimedes repeated use of the method of exhaustion that he refers to making approximations "as close as we please." Since Euclid uses the method in his Book XII, we would expect him to address this in some way in the *Elements*. In fact, in Proposition 1 of Book X Euclid proves the following.

Proposition 1. Two unequal magnitudes being set out, if from the greater there is subtracted a magnitude greater than its half, and from that which is left a magnitude greater than its half, and if this process is repeated continually, then there will be left some magnitude less than the lesser magnitude set out. And the theorem can similarly be proven even if the parts subtracted are halves.

The idea is that one of the two unequal magnitudes is "as small as we please." The repeated process of removing parts of size "at least half" allows us to chip away at the larger quantity until there is an amount as small as we please left. When using an ε argument for the convergence of a sequence in analysis, a similar approach is taken. The commonly named "Archimedean Principle" in the real numbers claims that if $a, b \in \mathbb{R}$ and $a > 0$, then there is a natural number $n \in \mathbb{N}$ such that $na > b$. This can be used to prove for any positive real number a and any real number b, there is a natural number n such that $b/n < a$. In this way, natural numbers can be used to produce values as small as we want (notice the b/n is between 0 and a; think of a as being the targeted small amount). See my online notes for Analysis 1 (MATH 4217/5217) on Section 1.3. The Completeness Axiom and notice Theorem 1-18 (The Archimedean Principle) and Corollary 1-18. Note 1.3.C of those notes also gives some history of the Archimedean Principle, which Archimedes gives as Assumption 5 in his *On the Sphere and Cylinder*, Book I.

Note 11.3.D. As an illustration of the method of exhaustion (well, *another* example following the many given in the Archimedes material) we go through the proof of Euclid's Proposition XII.2 of Book XII of *Elements*. Since Euclid predates Archimedes, Euclid's use of the method of exhaustion gives an example closer to the "genesis" of the idea. Recall the result from *Elements* is:

Proposition XII.2. Circles are to one another as the squares on their diameters. [That is, the area of a circle is proportional to the square of the diameter, and hence also proportional to the square of the radius.]

The constant of proportionality of the area of the circle to the square of its radius is the very definition of π. We follow the presentation given in Eves' textbook. Let A_1 and A_2 denote the areas of the circles and let their respective diameters be d_1 and d_2. The claim is then $A_1 : A_2 = d_1^2 : d_2^2$. First, we show that the difference between the area of the circle and an inscribed regular polygon can be made as small as we wish. Let AB be one side of a regular polygon (of at least three sides) inscribed in the circle and let M be the midpoint of the arc AB along the circle; see Figure 96 below.

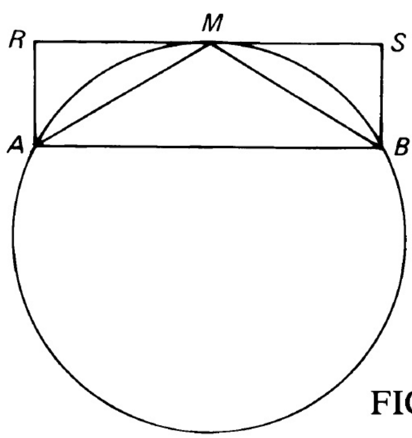
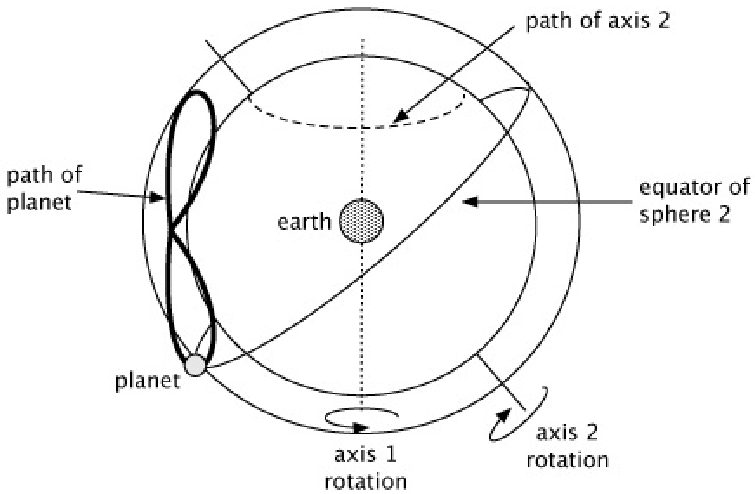


FIGURE 96

Introduce rectangle $ARSB$ as shown. The area of triangle AMB is half of the area of rectangle $ARSB$ and is greater then half of the area of the circular segment AMB (i.e., the intersection of the interior of the circle and the interior the rectangle; apologies to those who are looking for a higher level of rigor and less dependence on pictures!). By doubling the number of sides of the inscribed polygon (in which case the side AB is replaced with the two sides AM and MB), we increase the area of the polygon by more than half the difference in area between the polygon and the circle (the difference including the circular segment AMB for this portion of the original polygon, to the difference including the smaller circular segments AM and MB). So by Proposition 1 of Note 11.3.C, this process of doubling the number of sides of the regular polygon can be repeated to make the difference between the area of the circle and the area of some many sided polygon "some magnitude less than the [very small] lesser magnitude set out." The proof depends on using proof by contradiction ("*reductio ad absurdum*") twice. We present one of the proofs, the other being very similar. Throughout, we represent areas of circle and polygons by their names. ASSUME the ratios satisfy $A_1 : A_2 > d_1^2 : d_2^2$. That is, $A_1/A_2 > d_1^2/d_2^2$, do that $A_1 > A_2 d_1^2/d_2^2$. Then with a "set out magnitude" (think $\varepsilon > 0$ here, as in Calculus 1) of size $A_1 - A_2 d_1^2/d_2 > 0$, we can find a regular polygon P_1 inscribed in A_1 such that $A_1 - P_1 < A_1 - (A_2 d_1^2/d_2^2)$ or $P_1 > A_2 d_1^2/d_2^2$ or $P_1/A_2 > d_1^2/d_2^2$ or (in the ratio notation) $P_1 : A_2 > d_1^2 : d_2^2$. Let P_2 be a regular polygon similar to P_1 (that is, with the same number of sides) and inscribed in the second circle. Then by Proposition XII.1 of the *Elements* ("similar polygons inscribed in circles are to one another as the squares on their diameters") we have $P_1 : P_2 = d_1^2 : d_2^2$. Since $P_1 : A_2 > d_1^2 : d_2^2$, this implies that $P_1 : A_2 > P_1 : P_2 = d_1^2 : d_2^2$, or $P_1/A_2 > P_1/P_2$ or $1/A_2 > 1/P_2$ or $A_2 < P_2$. But P_2 is a regular polygon *inscribed* in the second circle, so this is a CONTRADICTION. So the assumption that $A_1 : A_2 > d_1^2 : d_2^2$ is false. We can similarly consider circumscribed regular polygons and also get a contradiction to the assumption that $A_1 : A_2 < d_1^2 : d_2^2$, thus concluding that $A_1 : A_2 = d_1^2 : d_2^2$, as claimed.

Note 11.3.E. Eudoxus attempted to explain the movements of the sun, moon, and planets relative to the "fixed" background stars with a purely geometric model of concentric spheres. His work was contained in *On speeds*, which is lost, but we know of it from comments of Aristotle (384 BCE–322 BCE) in his *Metaphysics* and Simplicius's (circa 480 CE–circa 540 CE) commentary on Aristotle's *De caelo*. In the 1870s, Giovanni Schiaparelli (March, 14 1835–July 4, 1910) worked out a "restoration" of Eudoxus' theory in investigated how accurate it could be made. His work appeared as "Die homocentrischen Sphären des Eudoxus, des Kallippus und des Aristoteles [The homocentric spheres of Eudoxus, Callippus and Aristotle]," *Abhandlungen zur Gesch. der Mathematik* **1**, 101–198 (1877). As an aside, Schiaparalli was an Italian astronomer and science historian. He made telescopic observations of Mars and classified surface features as seas, continents, and channels (in Italian, *canali*). The term was mistranslated into English as "canals," and this led to the speculative idea that Mars was covered by a system of canals constructed by a Martian civilization. American astronomer Percival Lowell (March 13, 1855–November 12, 1916) was fascinated by this idea and spent about 15 years observing Mars in Flagstaff, Arizona trying to map the canals; there *are* some erosional features in the form of valleys, but modern observations reveal that a canal system is nonexistent. Returning to Eudoxus' model, it is based on the assumption that the movements can be explained purely in terms of circular motion (an idea also assumed by Copernicus and, initially, by Kepler). Eudoxus places the planets on the equators of spheres which rotate and all such spheres have a common center (they are "homocentric" and the center is assumed to be the center of the stationary Earth). The spheres rotate uniformly. However, the outer planets (Mars, Jupiter, and Saturn) sometimes appear to move backward ("retrograde") in the night sky (as the Earth overtakes them in its faster revolution of the sun). The inner planets (Mercury and Venus) also display a retrograde motion, but it is the result of these planets overtaking the Earth in their faster revolution of the sun. So a single rotating sphere cannot explain the movements of the planets. To deal with this, Eudoxus placed the poles of the sphere containing a planet on a second larger rotating sphere (also with center at the center of the Earth). The poles of the second sphere were also placed on a third sphere concentric with and larger than the first two spheres, and the poles of the third sphere were similarly placed on a larger fourth sphere. That is, the motion of each planet required four rotating spheres. One of these spheres was needed to give the daily motion that *really* results from the rotation of the Earth. By choosing different rates of rotation, the model could be made to somewhat agree with observations. The sun and moon only required three spheres to explain their motion. Eudoxus did not bother to address the material from which the spheres were made or how they are mechanically connected in terms of the poles of some spheres being *on* other rotating spheres. David McClung's webpage on Eudoxus (accessed 5/5/2024) includes an image that illustrates how two spheres can interact (see figure below). This webpage also addresses the accuracy of Eudoxus' model. Two criticisms are raised.

First, retrograde loops of the outer planets are always the same size in the model, but this is not the case in reality (because the orbits of each planet, as well as the orbit of the Earth, are ellipses and not circles). Second, Eudoxus' model has the planets always the same distance from the center of the Earth so that they should always be the same brightness, however the planets vary in brightness (some, like Venus, dramatically so).



Some animations of Eudoxus' model are available on Henry Mendell's Eudoxos of Knidos (Eudoxus of Cnidus): astronomy and homocentric spheres webpage (accessed 5/9/2024). Notice in the third video how the figure-eight in the figure above is used to produce retrograde motion through time. A new interpretation of Eudoxus' model was given in Ido Yavetz's "On the Homocentric Spheres of Eudoxus," *Archives of the History of Exact Sciences*, **52**, 221–278 (1998). It is available on JSTOR (accessed 5/4/2024). It gives an alternative interpretation of Eudoxus' model to that given by Schiaparalli, which is also consistent with the known Greek astronomy of Eudoxus' time.

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