11.4. Archimedes' Method of Equilibrium

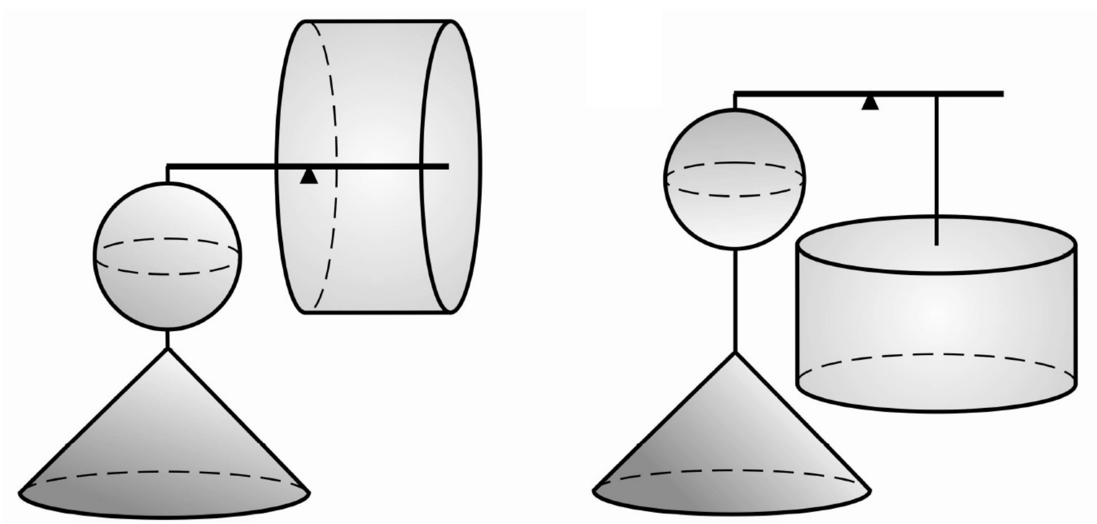
Note. We explored Archimedes' concept of balancing areas and line segments in Supplement. Archimedes' Method, Part 2. In particular, in that section we saw such an argument for a special case of Proposition 1 form Archimedes' *Method* on the area of a segment of a parabola bounded by a line (see Note AM2.C). We also saw applications of this idea in Supplement. The Content of Archimedes' Work, Part 1 when considering Propositions 1–17 of *Quadrature of the Parabola* (see Note AW.C). In this section, we consider Proposition 2 of the *Method* and give a slightly different argument based on Eves' presentation.

Note 11.4.A. We quote the statement of Proposition 2 from Thomas Heath's *The Works of Archimedes* (Cambridge University Press, 1897), the Supplement on "The *Method* of Archimedes" from 1911 (which appears in the Dover Publications version, 2002).

Proposition 2. (1) Any sphere is (in respect of solid content) four times the cone with base equal to a great circle of the sphere and height equal to its radius [or, equivalently, is two times the cone with base equal to a great circle of the sphere and height eual to it diameter]; and (2) the cylinder with base equal to a great circle of the sphere and height equal to the diameter is $1\frac{1}{2}$ times the sphere.

In other words, the volume of a sphere of radius r is four times the volume of the cone with radius r and height $h = r$. Since the cone has volume $\frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^3$ so that the volume of the sphere is the $\frac{4}{3}\pi r^3$ (as expected). Since the cylinder has radius r and height $h = 2r$ then the cylinder has volume $\pi r^2 h = \pi r^2(2r) = 2\pi r^3$.

This is claimed to be $1\frac{1}{2}$ times the volume of the sphere, so the sphere has volume $\frac{2}{3}(2\pi r^3) = \frac{4}{3}\pi r^3$, again. This is Archimedes famous formula expressing the volume of a sphere as 2/3 of the volume of a circumscribed cylinder. Archimedes considers balancing cross sections of the sphere, cone, and cylinder. With these cross section "taken together," he concludes that the solids themselves similarly balance. This proposition (along with Propositions 1 and 5) is presented in the 46 page booklet *The Illustrated Method of Archimedes: Utilizing the Law of the Lever to Calculate Areas, Volumes, and Centers of Gravity* by Andre Assis and C.P. Magnaghi (C. Roy Keys Inc., 2012). Part of the image on the cover illustrates Archimedes' concept of balancing solids.



Part of the cover of *The Illustrated Method of Archimedes* by Assis and Maganghi

Note 11.4.B. We now give the argument given by Eves on page 384. It is similar to the argument of Archimedes given in the *Method* (though it involves "Δx slices" as opposed to cross sections), but it a bit more modern uses a notation you are familiar with from Calculus 1. We introduce an xy coordinate system with the point N on the sphere of radius r at the origin. Let the cone of radius r and height $2r$ be circumscribed about the sphere so that the bases of the cylinder are circles intersecting the xy plane at right angles to the xy plane along the line segment AN (for the left base) and along the line segment BS (for the right base). Also introduce a cone with vertex at the origin, axis lying along the positive x axis, and with base of radius $2r$ intersecting the xy plane along line segment CS so that the height is $2r$ (see Figure 98 below). Notice that this is *not* the cone referenced in Proposition 2(1) since the height of this cone is $2r$, whereas the height of the cone in Proposition 2(1) is r (but the volumes of these two cylinders is related by a factor of 2). The equation of the circle of intersection of the sphere with the xy plane is $(x-r)^2+y^2=r^2$ so that $y^2=r^2-(x-r)^2=2xr-x^2=x(2r-x)$ (this will represent the radius of a cross section). Take a "Δx slice" at variable coordinate x of the (a) sphere, (b) cylinder, and (c) cone. With Δx as the thickness of the slices, they have approximate volumes of (a) $\pi x(2r-x)\,\Delta x$, (b) $\pi r^2\,\Delta x$, and (c) $\pi x^2\,\Delta x$, respectively.

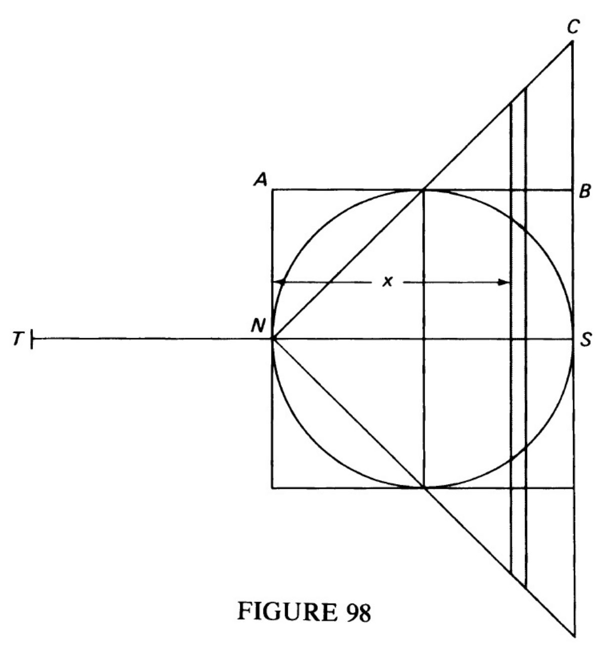


FIGURE 98

Next, introduce point T on the negative x axis at a distance of $2r$ from the origin (then the coordinates of T are $(-2r, 0)$). If the slices of the sphere and cone are supported at point T then the moment about point N of these combined slices (that is, the volume of the slices times the distance of point T from the fulcrum at point N) is (well, approximately):

$$(\pi x(2r - x)\,\Delta x + \pi x^2\,\Delta x)(2r) = (2r\pi x - \pi x^2 + \pi x^2)(2r\,\Delta x) = 4\pi r^2 x\,\Delta x. \qquad (*)$$

If the slice of the cylinder are supported at point T then the moment about the point N is $(\pi r^2\,\Delta x)(x) = \pi r^2 x\,\Delta x$. To get the slices of the sphere and cone to balance the slices of the cylinder, we need to move the cylinder $4r$ units to the right, as illustrated above, not to scale, in the figure from the cover of *The Illustrated Method of Archimedes* (right). Notice in the figure that the solids are suspended over their centers of mass. However, Eves simply substitutes the volume of the slice of the cone into equation $(*)$ to get:

$$(\text{slice of sphere} + \text{slice of cone})(2r) \approx 4r(\text{slice of cylinder}).$$

This requires an approximation since the use of Δx implies an approximation of the volume of the slices (the edges of which are curved in the cases of the sphere and cone). "Adding a large number of these slices together" (as Eves says on page 384), gives (*approximately*):

$$2r(\text{volume of sphere} + \text{volume of cone}) = 4r(\text{volume of cylinder}).$$

We already know the volumes of a cone and cylinder. Here the cone has volume $\frac{1}{3}\pi(2r)^2(2r) = \frac{8}{3}\pi r^3$ and the volume of the cylinder is $\pi r^2(2r) = 2\pi r^3$. So we now have

$$(\text{volume of sphere} + \text{volume of cone}) = 2(\text{volume of cylinder})$$

$$\text{or volume of sphere} = 2(\text{volume of cylinder}) - \text{volume of cone}$$

$$\text{or volume of sphere} = 2(2\pi r^3) - \frac{8}{3}\pi r^3 = \frac{4}{3}\pi r^3.$$

The expression of the volume of the sphere in terms of the cone and cylinder, as stated in Proposition 2, now follows.

Note 11.4.C. In Archimedes approach, he considers cross sections and "weighs" these sections by their areas. In this way, the balancing of the sections does not need any approximation. However, Archimedes avoids summing the cross sections and instead considers the cross sections "taken together." He assumes that the balancing property satisfied by the cross sections is also satisfied by the solids themselves. This occurs repeatedly in the *Method*. See, for example, Note AM2.D of Supplement. Archimedes' *Method*, Part 2 on the area bounded by a segment of a parabola. In Calculus 1, such a problem would be approached by partitioning up the diameter NS of the sphere of Figure 98 as $P = \{n_0 = N, x_1, x_2, \ldots, x_n = S\}$, introducing the widths $\Delta x_i = x_i - x_{i-1}$, and considering the resulting slices of the sphere, cylinder, and cone. A Riemann sum based on the slices would be introduced, in this case representing the moments about point N of the slices. A limit as the norm of the partition, $\|P\| = \max 1 \le i \le n \Delta x_i$, approaches zero then produces the precise value of the moments of the solids in terms of an integral. See my online notes for Calculus 1 (MATH 1910) on Section 5.3. The Definite Integral for relate details and definitions. Often times a *regular partition* (for which $\|P\| = \Delta x_i$ for all $1 \le i \le n$) is used in specific examples in Calculus 1; see Example 5.2.5 of Section 5.2. Sigma Notation and Limits of Finite Sums where $\int_0^1 (1 - x^2)\, dx$ is evaluated in exactly this way. Interestingly, Archimedes also employs such a technique in some of his proofs in *Quadrature of the Parabola*; see Note AW.C of Supplement. The Content of Archimedes' Work, Part 1 (he combines this with his physical "balancing" technique). He even comes close to taking a limit as the number of slices determined by the regular partition approaches infinity! Clever chap, that Archimedes...

Note 11.4.D. Archimedes' arguments based on physical reasoning are novel and, arguably, not mathematically rigorous. However, they do give insight to *how* he *thought* about these problems (that is the real revelation of the discovery of the *Method* and its analysis by Heiberg in 1906, and the rediscovery of it an new analysis in the 21st century, as discussed in Supplement. Archimedes' Method, Part 1)! Archimedes was compelled to give a less physical and more traditional argument for his claim that the volume of a sphere is $\frac{4}{3}\pi r^3$. This he did in *Sphere and Cylinder* Book I when he gave a method of exhaustion proof of the above claim (in Proposition 34 and its corollary). See also Note AW.D of Supplement. The Content of Archimedes' Work, Part 1.

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