14.2. Impossibility of Solving the Three Famous Problems with Euclidean Tools

Note. In this section, we further explore "The Three Famous Problems" of Section 4.3. In Chapter 4, "Duplication, Trisection, and Quadrature," we explored attempts to give compass and straight edge constructions of the (1) duplication of the cube (in Section 4.5), (2) trisection of an angle (in Section 4.6), and (3) quadrature of the circle (or "squaring the circle"; in Section 4.7). None of these constructions were successful, but we saw how certain curves could be used to solve these problems (such as conic sections, the conchoid of Nicomedes, and the quadratrix of Hippias; none of these curves are constructible with a compass and straight edge). We also some mechanical devices that could be used to "solve" the three famous problems. A proof that none of the three famous problems can be solved with a compass and straight edge alone requires some knowledge of modern algebra and the properties of fields. As a consequence, this proof of the impossibility of such a construction was not given until the 19th century.

Note. We start with a definition. A real number is an *algebraic* number if it is a root of some polynomial with rational coefficients (we could also state that an algebraic real number is a root of some polynomial with integer coefficients since, when setting a rational polynomial equal to 0, we can multiply through by a common multiple of the denominators of the coefficients to produce an equivalent polynomial equation where the coefficients are integers). Every rational number is algebraic since $x = p/q \in \mathbb{Q}$, where $p, q \in \mathbb{Z}$ and $q \neq 0$, is a root of the first degree polynomial $qx - p$. The nth root of every rational number is algebraic since $x = \sqrt[n]{p/q}$ is a root of the n degree polynomial $qx^n - p$. In particular, $\sqrt{2}$ is algebraic though, as shown by the Pythagoreans, it is not rational; see Note 3.5.B of Section 3.5. Discovery of Irrational Magnitudes. A real number is *transcendental* if it is not algebraic. With $\mathbb{A}$ denoting the algebraic numbers (so that $\mathbb{T} = \mathbb{R} \setminus \mathbb{A}$ denotes the transcendental numbers; this is not a standard notation), we have the relationships:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{Q} \subset \mathbb{A} \subset \mathbb{R}, \text{ and } \mathbb{A} \cup \mathbb{T} = \mathbb{R}.$$

Each subset inclusion here is proper.

Note. Eves (on page 540) states that the following two theorems "are established":

1. The magnitude of any length constructible with Euclidean tools from a given unit length is an algebraic number.

2. From a given unit length it is impossible to construct with Euclidean tools a segment the magnitude of whose length is a root of a cubic equation with rational coefficients but with no rational root.

Eves then dispatches with the construction problems. First he observes that quadrature of the circle is impossible with a compass and straight edge because this requires construction of $\sqrt{\pi}$, which is transcendental, in violation of the first theorem. Next, duplication of the cube with Euclidean tools is impossible because this requires the construction of $\sqrt[3]{2}$, in violation of the second theorem. Finally, the general trisection of an angle with Euclidean tools is impossible because the particular angle $60°$ cannot be trisected, since this requires solving the polynomial equation $8x^3 - 6x - 1 = 0$, in violation of the second theorem (we need $x = \cos(60°/3) = \cos 20°$ here; see Note 4.6.A of Section 4.6. Trisection of an Angle). We will give references from Introduction to Modern Algebra 2 (MATH 4137/5137) to elaborate on Eves' two theorems and spell out the history of the results in more detail.

Note. With the claim that $\sqrt{\pi}$ is transcendental, we have that π itself must be transcendental (because squares and square roots of algebraic numbers are algebraic). The fact that transcendental numbers exist may, itself, be surprising! Another famous transcendental number is the base of the natural logarithm function, e. Since $\sqrt[3]{2}$ is algebraic but is not constructible (by Eves' second theorem), we see that the set of *constructible numbers*, which we denote C (the font "$\mathbb{C}$" is almost universally used to denote the complex numbers), is a superset of the rational numbers and a subset of the algebraic numbers. So we can extend the above subset inclusion as follows:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{Q} \subset C \subset \mathbb{A} \subset \mathbb{R}.$$

Again, each subset inclusion is proper.

Note. We now state some definitions and observations from Introduction to Modern Algebra 1 (MATH 4127/5127). See my online notes for that class on Section IV.18. Rings and Fields for more details. The numbering scheme used here is the same as in those online notes.

Definition 18.1. A *ring* $\langle R, +, \cdot \rangle$ is a set R together with two binary operations $+$ and $\cdot$, called *addition* and *multiplication*, respectively, defined on R such that:

$\mathcal{R}_1$**:** $\langle R, + \rangle$ is an abelian group (that is, addition is associative, commutative, and there is an additive identity and additive inverses).

$\mathcal{R}_2$**:** Multiplication $\cdot$ is associative: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in R$.

$\mathcal{R}_3$**:** For all $a, b, c \in R$, the *left distribution law* $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ and the *right distribution law* $(a + b) \cdot c = (a \cdot c) + (b \cdot c)$ hold.

Note. We do not require commutativity of multiplication in a ring. If it is present, then we have a *commutative ring.* There may not exist a multiplicative identity either. A ring which does have a multiplicative identity is a *ring with unity* (unity is denoted "1"). You can see how this algebraic structure is taking on rather abstract properties! To make things more tangible, we consider an example of a noncommutative ring. Consider the ring $M_n(\mathbb{R})$ of all $n \times n$ matrices with real entries. Properties $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}_3$ are familiar to you from Linear Algebra (MATH 2010). However, you know that matrix multiplication is not, in general, commutative. Also, many square matrices do not have multiplicative inverses (namely, the *singular* matrices), so the desirable property of the existence of (multiplicative) inverses may not be present in a ring. A ring in which every nonzero (i.e., non-additive-identity element) has a multiplicative inverse is called a *division ring.* A quick word of caution. In a sense, there is "no such thing" as division! There is *multiplication* and *multiplicative inverses*, but no division. We would denote the multiplicative inverse of ring element a as a^{-1}. Then we read ba^{-1} as "b times a inverse," NOT as "b divided by a" (if you want to press this, you could also claim that "times" should be replaced with "multiplied on the right by"…).

Note. For ring R with additive identity denoted "0", we have for all $a, b \in R$ that (1) $0a = a0 = 0$, (2) $a(-b) = (-a)b = -(ab)$, and (3) $(-a)(-b) = ab$ (this is Theorem 18.8 in the Modern Algebra notes). So in a ring, we have the familiar behavior of multiplication in terms of an interaction between "positive" and "negative" elements. However, these concepts are largely restricted to real numbers! For example, there are no such things as "positive matrices" (you may have heard of "positive definite matrices," but that is something different). In fact, there is no such thing as positive or negative complex numbers (beyond the fact that the complex numbers contain the real numbers, and the real numbers can be expressed as positive, negative, or 0). So in the setting of a ring, the third property "$(-a)(-b) = ab$" should not be read as "a negative times a negative is positive," but instead as "the additive inverse of a times the additive inverse of b equal a times b" (since the symbol "$-$" denotes an additive inverse). This is related to the fact that, similar to the observations about "division" in the previous note, there is so such thing as subtraction, but instead there is addition of additive inverses). This abstraction which may at first seem awkward and ugly, it is the key to proving the impossibility of solving the Three Famous Problems with Euclidean Tools!

Definition 18.16. A *field* is a commutative division ring.

Note. Fields have the algebraic structure we want (we'll see that the constructible numbers C form a field)! Examples of fields are $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$, where we use the usual addition and multiplication. An example of a *finite field* is the integers modulo prime p, $\mathbb{Z}_p$; it is necessary to use a prime number to insure the existence of multiplicative inverses.

Note. I have an online video, "Compass and Straight Edge Constructions" posted on YouTube, as a supplement to my Introduction to Modern Algebra 2 notes on Section VI.32. Geometric Constructions . A transcript of the video is available in PDF. The rules by which compass and straight edge constructions are performed are given in the video as follows. We start with a line segment of a length which we define as the unit length. For example, we could assume the line segment lies along the x-axis between the points $(0, 0)$ and $(1, 0)$. We can construct other lines or line segments using a straight edge through two constructed points. Given a point p and a line segment of a given length ℓ, we can use the compass to construct a circle with center p and radius ℓ. Given a line segment of a certain length, a line segment of the same length can be constructed on any given line. A point is *constructed* when it results from the intersection of two lines, two circles, or a line and a circle.

Definition. A real number α is *constructible* if we can construct a line segment of length $|\alpha|$ in a finite number of steps from a given unit length segment and a compass and straight edge (as described above).

Note. The relevant three key results from modern algebra are the following.

Theorem 32.1. If α and β are constructible real numbers, then so are $\alpha+\beta$, $\alpha-\beta$, $\alpha\beta$, and α/β if $\beta \neq 0$.

Corollary 32.5. The set of constructible real numbers C forms a subfield of the field of real numbers.

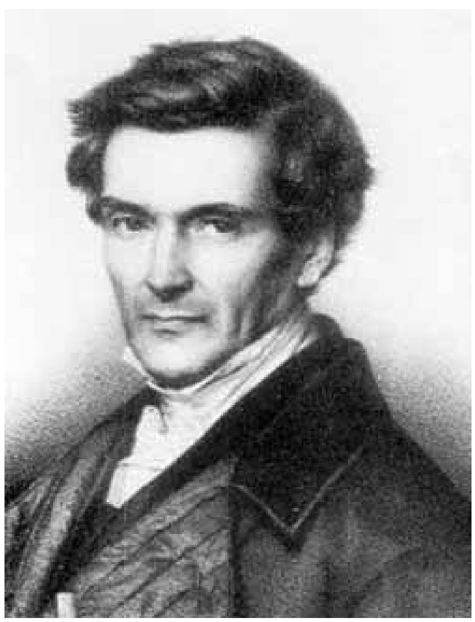
Theorem 32.6. The field of constructible real numbers consists precisely of all real numbers that we can obtain from $\mathbb{Q}$ by taking square roots of positive numbers a finite number of times and applying a finite number of field operations (the field operations are addition and multiplication).

Note. To show the impossibility of solving the Three Famous Problems, we need a clear explanation of why $\sqrt[3]{2}$, a real solution of $8x^3-6x-1=0$, and π are not constructible. This requires some knowledge of the degree of a field extension (the needed result is Corollary 32.8 in Section VI.32. Geometric Constructions). A more rigorous exploration of these ideas are also given in graduate-level Modern Algebra 1 (MATH 5410) in Section V.1.Appendix. Ruler and Compass Constructions.

Note. We now turn to the history of the above mentioned material. My choice for a text book in Introduction to Modern Algebra 1 and 2 (MATH 4127/5127 and 4137/5137) is John B. Fraleigh's *A First Course in Abstract Algebra*, Seventh Edition, Pearson Education (2003). This book includes brief historical notes by Victor Katz (author of *History of Mathematics, An Introduction*, 3rd edition, Addison-

Wesley, 2009; I have planned online notes based on his book as a possible source in this class). In Fraleigh's Section VI.32. Geometric Constructions, it is stated on page 298 that "Peirre Wantzel (1814–1848)...proved Corollary 32.8 [on the degree of a field extension] and also demonstrated Theorems 32.9 [on the impossibility of the duplication of the cube] and 32.11 [on the impossibility of trisection of an angle]." It is surprising that Wantzel does not even appear in Eves' book!

Note. Pierre Wantzel was born in Paris in 1814. At the age of 14 he entered the Collège Charlemagne and at age 15 edited a second edition of Antoine Reynaud's book *Treatise on Arithmetic*. He entered the engineering school of Ponts et Chaussées in 1834 and became a lecturer at the prestigious École Polytechnique in 1838. From 1841 he was professor of applied mechanics at the École des Ponts et Chaussées.



Pierre Laurent Wantzel (June 5, 1814–May 21, 1848)

The image of Wantzel and this historical information is from the MacTutor History of Mathematics Archive biography of Wantzel (accessed 4/30/2023). Little information seems to be available on his early death at the age of 33, but in Florian Cajori's "Pierre Laurent Wantzel," *Bulletin of the American Mathematical Society*, **24**(7), 339–347 (1918), Jean Claude Saint-Venant (who wrote "Biographie: Wantzel," *Nouvelles Annales de Mathématiques (Terquem et Gerono)* **7**, 321–331, (1848) upon Wantzel's death) is quoted as saying of Wantzel:

> "... Ordinarily he worked evenings, not lying down until late; then read, and took only a few hours of troubled sleep, making alternately wrong use of coffee and opium, and taking his meals at irregular hours until he was married. He put unlimited trust in his constitution, very strong by nature, which he taunted at pleasure by all sorts of abuse. He brought sadness to those who mourn his premature death."

Wantzel published over 20 mathematical papers (they are listed in Saint-Venant's biography of Wantzel).

Note. Wantzel's paper that proves the impossibility of the duplication of the cube and the trisection of an angle is: "Recherches sur le moyens de reconnaitre si un Problème de Géométrie peut se résoudre avec la règle et le compas" [Research on Ways to Recognize if a Problem of Geometry can be Solved with Ruler and Compass], *Journal de Mathématiques pures et appliquées*, **2**, 366–372 (1837). One would assume that a rigorous resolution of these 2000-plus years old construction problems would meet with great attention and fame. This was not the case. Jesper Lützen adresses this in "Why was Wantzel overlooked for a century? The changing importance of an impossibility result," *Historia Mathematica*, **36**(4), 374–394 (2009):

> "Though the classical construction problems were not at the center stage of mathematical research during the early 19th century, they were certainly well known. The quadrature of the circle was the most celebrated of the problems, but the duplication of the cube and the trisection of the angle enjoyed so much fame that one would have expected Wantzel's resolution of them to have made an impression on the mathematical community. However, that did not happen until a century later when Wantzel began to be generally acknowledged as the first person to have solved the problems."

Lützen speculates that Cajori (in his 1918 biography of Wantzel that appear in the *Bulletin of the AMS*) may be responsible for the spread of Wantzel's fame in the 20th century.

Note. Another compass and straight edge construction associated with Wantzel is the construction of regular polygons. Carl Friedrich Gauss (April 30, 1777–February 23, 1855) gave a construction using Euclidean tools of a 17 sided regular polygon in 1796. In his *Disquisitiones Arithmeticae* of 1801, Gauss showed that a regular n-gon can be constructed if n is a power of 2 or if n is the product of a power of 2 and any number of distinct "Fermat primes." A *Fermat prime* is a prime number of the form $2^{(2^m)}+1$ (the only known Fermat primes are 3, 5, 17, 257, and 65537; this shows how Gauss' construction of a 17 sided polygon fits into this scheme). More on Fermat primes is in my online notes for Mathematical Reasoning (MATH 3000) on Section 6.7. More on Prime Numbers . However, Gauss did not show that the given values of n are are a necessary condition for the construction of a regular n-gon. As was a tradition for Gauss, he claimed that the condition was necessary, but he failed to present a proof. The necessity of the condition is also given in Wantzel's 1837 paper. Therefore the "Gauss-Wantzel Theorem" (as it is called on the Wikipedia page on Constructible Polygons; accessed 4/30/2023) can be stated as:

The Gauss-Wantzel Theorem. A regular n-gon is constructible with a compass and straight edge if and only if $n = 2^k p_1 p_2 \cdots p_t$ where k and t are nonnegative integers, and $p_1, p_2, \ldots, p_t$ are distinct Fermat primes.

This story of the construction of regular n-gons is told in Ian Stewart's *Why Beauty Is Truth: A History of Symmetry*. Basic Books (2007); see pages 134 to 136. This book seems unique in that it is a "popular level" book that includes a small biography of Pierre Wantzel (see pages 126 and 127).

Note. By Theorem 32.6 stated above (or by Eves' first theorem above), all constructible numbers are algebraic. Quadrature of the circle requires that π is constructible. Therefore, if we can show that π is transcendental (that is, not algebraic) then it is not constructible and the impossibility of the quadrature of the circle follows. This is how the impossibility of the construction was shown in 1882 by Ferdinand von Lindemann (April 12, 1852–March 6, 1939).

Lindemann was born in Hanover (in modern-day Germany). He studied in Göttingen, Munich, and Erlangen. He wrote his dissertation under the direction of the geometer Felix Klein (April 25, 1849–June 22, 1925). He taught in England and France, and back in Germany he taught at the University of Freiburg, University of Königsberg, and the University of Munich. In 1873 Charles Hermite (December 24, 1822–January 14, 1901) proved that e is transcendental. Shortly after, Lindemann visited him to discuss the proof. Using methods similar to those of Hermite, Lindemann proved in 1882 that π is transcendental and published it in *Über die Zahl* π [On the Number π], *Mathematische Annalen*, **20**, 213–225 (1882). A copy (in German) can be viewed online on the Goettingen Digitization Center webpage (accessed 4/30/2023). In the book by Edward Berger and Robert Tubbs, *Making Transcendence Transparent: An Intuitive Approach to Classical Transcendental Number Theory*, Springer (2004), a proof of the transcendental nature of π is given (see Chapter 3, Theorem 3.1, and Corollary 3.2).

Note. We now discuss some other relevant results on e, π, irrational numbers, and transcendental numbers. These observations are largely based on the Wikipedia page on Transcendental Numbers (accessed 4/30/2023). Johann Lambert (August 26/28, 1728–September 25, 1777) conjectured that e and π were transcendental numbers and proved that π is irrational (and sketched a proof that π is transcendental) in "Mémoire sur quelques propriétés remarquables des quantités transcendantes, circulaires et logarithmiques," *Mémoires de l'Académie Royale des Sciences de Berlin*, 265-322 (1768). Joseph Liouville (March 24, 1809–September 8, 1882) was the first to prove the existence of transcendental numbers in 1844, presented *Liouville's constant*, $\sum_{n=1}^{\infty} 10^{-n!}$, in 1851 and proved that it is transcendental. As mentioned above, Hermite proved that e is transcendental in 1873 and Lindemann proved that π is transcendental in 1882. In terms of cardinality, the algebraic numbers are countable, as shown by Georg Cantor (March 3, 1845–January 6, 1918) in 1874 (you might see this result in Analysis 1 [MATH 4217/5217]; it follows from Exercise 1.3.14 of James Kirkwood's *An Introduction to Analysis*, Third Edition, CRC Press [2021] and I have online notes for Kirkwood's Section 1.3. The Completeness Axiom). Since the real numbers are uncountable, then it follows that the transcendental numbers are uncountable. More on the cardinalities of infinites sets can be found in my online notes for Mathematical Reasoning (MATH 3000) on Section 4.3. Countable and Uncountable Sets and Section 4.4. More on Infinity.

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