2.4. Babylonia: Geometry

Note. In this section, we consider some of the geometric problems addressed in the Babylonian mathematical clay tablets. We mention areas and volumes of geometric objects (though this could also be considered algebra), and consider the Pythagorean Theorem and estimates of π.

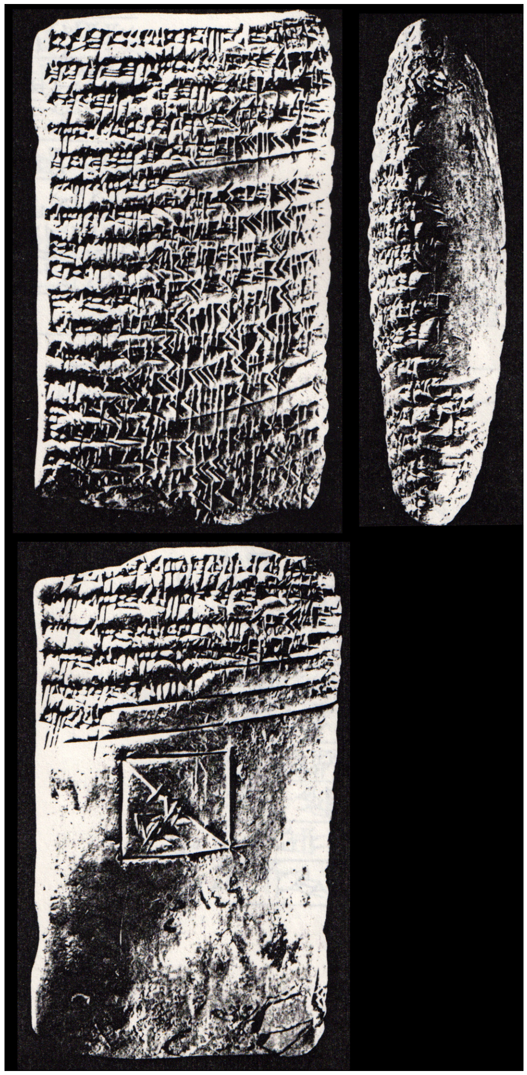
Note. In the Babylonian cuneiform clay tablets, described in Section 1.4. Simple Grouping Systems, there are sufficient examples to suggest that the Babylonians must have been familiar with the general rules for the area of rectangle, area of right and isosceles triangles, area of trapezoid having one side perpendicular to the parallel sides, volume of a rectangular parallelepiped, and the volume of a right prism with a special trapezoidal base. They knew that a perpendicular through the vertex of an isosceles triangle bisects the base, and angles inscribed in a semicircle are right angles. See Eves, page 42.

Note 2.4.A. Otto Neugebauer, who we met in Section 2.2. Babylonia: Sources, says in his *The Exact Sciences in Antiquity*, second edition (Brown University Press, 1957):

> "The great majority of mathematical texts [i.e., clay tablets of mathematical content] are 'Old Babylonian'; that is to say, they are contemporary with the Hannurapi [i.e., King Hammurabi] dynasty, thus roughly belonging to the period from 1800 to 1600 B.C. The second, and much smaller group of 'Seleucid', i.e. datable to the last three centuries B.C. . . . The only essential progress which was made [between these two periods] consists in the use of the 'zero' sign in the Seleucid texts. . . It seems plausible that the expansion of numerical procedures is related to the development of a mathematical astronomy in this latest phase of Mesopotamian science." (See Neugebauer's page 29.)

He classifies the mathematical tablets as falling into two categories: "table texts" and "problem texts." A large percentage of the table texts are exercises written by apprentice scribes which Neugebauer calls "school texts." Babylonian mathematics was more developed in the algebraic and computational areas, with geometry playing a less significant role (see Neugebauer, page 44). In fact, the geometry they addressed was mostly related to areas and volumes of triangles, trapezoids, cones, and pyramids in terms of lengths of sides of the objects; we call these ideas "geometric algebra" in Section 3.6. Algebraic Identities.

Note 2.4.B. In 1962 a clay tablet was found in northeastern Iraq in the ruins of ancient Eshnunna. Today it is in the National Museum of Iraq with Museum Number IM 067118 (it is also known as Db_2 146). This tablet contains a proof of the Pythagorean Theorem in the context of finding the dimensions of a rectangle. An image of the tablet is given below.

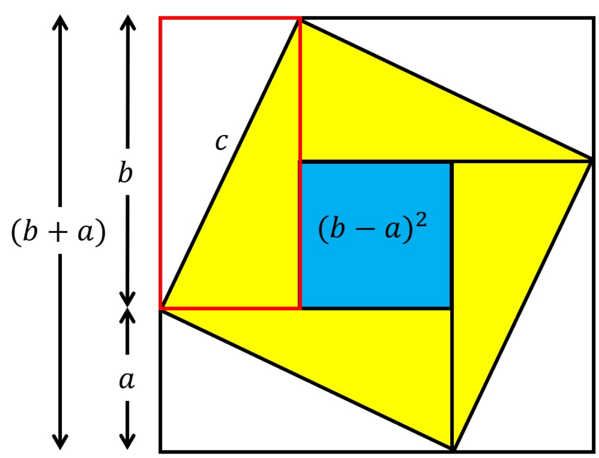


From the Cuneiform Digital Library Initiative (accessed 8/5/2023).

The problem on the tablet is described by Peter Rudman in his *How Mathematics Happened: The First 50,000 Years* (Prometheus Books, 2007) as follows (pages 236 and 237): "The area of a rectangle if 0.45_{60} and its diagonal is 1.15_{60}. Find its length and width." Rudman is expressing numbers base 60. Now $0.45_{60} = 0.75_{10} = 0.75$ and $1.15_{60} = 1.25_{10} = 1.25$.. With the length and width as a and b, and c as the diagonal, the problem translates into our modern notation as: "$ab = 0.75$ and $c = 1.25$ implies $a =?$ and $b =?$" As Rudman explains, the solution is given by the sequence of steps:

	Babylonian Algorithm	**Algebraic Generalization**
Step 1	$2 \times 0.75 = 1.5$	$2ab$
Step 2	$3(1.25)^2 = 1.5625$	c^2
Step 3	$1.5625 - 1.5 = 0.0625$	$(b-a)^2 = c^2 - 2ab$
Step 4	$\sqrt{0.0625} = 0.25$	$(b-a) = \sqrt{c^2 - 2ab}$
Step 5	$0.25/2 = 0.125$	$(b-a)/2 = \sqrt{c^2 - 2ab}/2$
Step 6	$(0.125)^2 = 0.015625$	$[(b-a)/2]^2 = (c^2 - 2ab)/4$
Step 7	$0.75 + 0.015625 = 0.765625$	$[(a+b)/2]^2 = (c^2 - 2ab)/4 + ab$
		$= (c^2 + 2ab)/4$
Step 8	$\sqrt{0.765625} = 0.875$	$(a+b)/2 = \sqrt{c^2 - 2ab}/2$
Step 9	$0.875 + 0.125 = 1$	$b = (a+b)/2 + (b-a)/2$
Step 10	$0.875 - 0.125 = 0.75$	$a = (a+b)/2 - (b-a)/2$

Notice that we have $(0.75)^2 + (1)^2 = (1.25)^2$, or $a^2 + b^2 = c^2$ as we would expect from the Pythagorean Theorem. However, we have not used the Pythagorean Theorem in the computation above. In the following figure, the rectangle with width a, height b, and diagonal c is outlined in red.



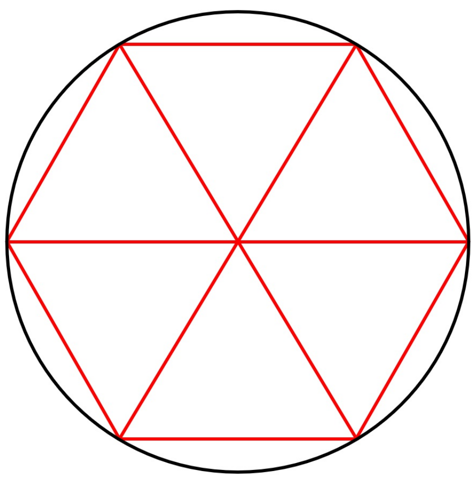
Based on Figure 5.4.4 of Rudman.

Notice that the square with side of length c has as its area the sum of the yellow areas ($4 \times (ab)/2 = 2ab$) and the blue area ($(b-a)^2$). In the algorithm, Steps 1 through 5 use the geometry to calculate $(b-a)/2$ in terms of the givens ab and c. The large square is of area $(a+b)^2$ and is composed of four white triangles (of total area $2ab$) and the square of area c^2. Steps 6 through 8 use this geometry to calculate $(a+b)/2$ in terms of the givens ab and c. Finally, Steps 9 and 10 combine $(b-a)/2$ and $(a+b)/2$ to give a and b. Implied by the given argument is two proofs of the Pythagorean Theorem! First, by Step 3 we have $(b-a)^2 = c^2 - 2ab$ (this results from considering the area of the square with side of length c). The Babylonians would also know that $(b-a)^2 = a^2 + b^2 - 2ab$, since this is easily established with geometric algebra (this is to be done in Problem Study 3.8(a)). The Pythagorean Theorem then follows: $a^2 + b^2 = c^2$. Notice that by considering the diagonal of a rectangle, a right triangle is automatically produced. Second, by Step 7, we have $(a+b)^2 = c^2 + 2ab$ (this results from considering the area of the large square with side of length $(a+b)$). The Babylonians would know that $(a+b)^2 = a^2 + b^2 + 2ab$, since this is easily established with geometric algebra (see Figure 16 in Section 3.6. Algebraic Identities). The Pythagorean Theorem then follows: $a^2 + b^2 = c^2$. We will see in Section 2.6. Babylonia: Plimpton 322 additional evidence that the Babylonians knew the Pythagorean Theorem. In that section, several *Pythagorean triples* are given; that is, natural numbers a, b, c such that $a^2 + b^2 = c^2$ (the easiest example begin $3^2 + 4^2 = 5^2$). However, since the Pythagorean triples are restricted to natural numbers, their study is insufficient for a proof of the Pythagorean Theorem. Since lengths of line segments can be any positive real number (Euclid will call these *magnitudes* in his study of proportions;

see Note 5.4.J in Section 5.4. Contents of the "Elements"), the argument given here covers the general case. Of course it is too early in the history of mathematics to call what the Babylonians have given as a "proof."

Note 2.4.C. For the Babylonian approximation of π, we refer to Eves' Problem Study 2.5, "The Susa Tablets." Parts (a) and (b) of this problem states (see Eves' page 59): "In 1936 a group of Old Babylonian tablets was lifted at Susa, about 200 miles from Babylon. One of the tablets compares the areas and the squares of the sides of regular polygons of 3, 4, 5, 6, and 7 sides. . . . On the same tablet. . . , the ratio of the perimeter of a regular hexagon to the circumference of the circumscribed circle is given as 0;57,36 [base 60]. Show that this leads to 3;7,30 [base 60] or $3\frac{1}{8}$ as an approximation of π."

The image above is from the Louvre Museum website. It shows clay tablet SB 13088, which was found in 1933 and is housed in the Louvre's Department of Oriental Antiquities. This appears to be the tablet referred to in Eves' problem (though there is the 1933/1936 date conflict). It shows a regular heptagon on one side (left) and a regular hexagon on the other (right). Other information on the tablet is online on Frank J. Swetz's "Mathematical Treasure: Cuneiform Tablet Depicting Heptagon," *Convergence* (July 2021). To solve Problem Study 2.5(b), we have in modern notation that the circumference of a circle is $2\pi r$. In the image below, the hexagon is broken into six equilateral triangles, so that the perimeter of the inscribed hexagon is $6r$. So the ratio of the perimeter of the hexagon to the circumference of the circle is $6r/2\pi r = 3/\pi$, which is given as a sexagesimal number as 0;57,36. This translates to $57/60 + 36/60^2 = 3456/3600$. We therefore have $3/\pi \approx 3456/3600$ or $10800/3456 = 3\frac{1}{8} \approx \pi$. To complete the problem, we convert $3\frac{1}{8} = 3.125$ to sexagesimal form. Since $0.1 = \frac{1}{10} = \frac{7}{60} \approx 0.1167 < \frac{1}{8} = 0.125 < \frac{8}{60} \approx 0.1333$ then the first digit of the sexagesimal form of $\frac{1}{8}$ if 7. Next, $\frac{1}{8} - \frac{7}{60} = \frac{1}{120} = \frac{30}{60^2}$ so that the second digit of the sexagesimal form is $\frac{1}{8}$ is 30, and hence $3\frac{1}{8} = 3;7,30$.



Note 2.4.D. The frustum (or as Eves calls it "frustrum") of a pyramid is the volume that results when the top of a pyramid is cut off, as illustrated below. Eves claims (on page 42) that in the discussion of the volumes of frustums of a pyramid,

a general cubic equation arises by considering the system of equations

$$z(x^2 + y^2) = A, \quad z = ay + b, \quad x = c.$$

His source on this is no doubt Raymond Archibald's "Babylonian Mathematics," *Isis*, **26**(1), 63–81 (1936). This can be viewed online on the JSTOR webpage (accessed 8/6/2023). Eves uses the exact same variables as Archibald; see Archibald's pages 77 and 78. Unfortunately, neither Eves nor Archibald give an details on what the equations represent. Archibald mentions that this appears on clay tablet YBC 4708.

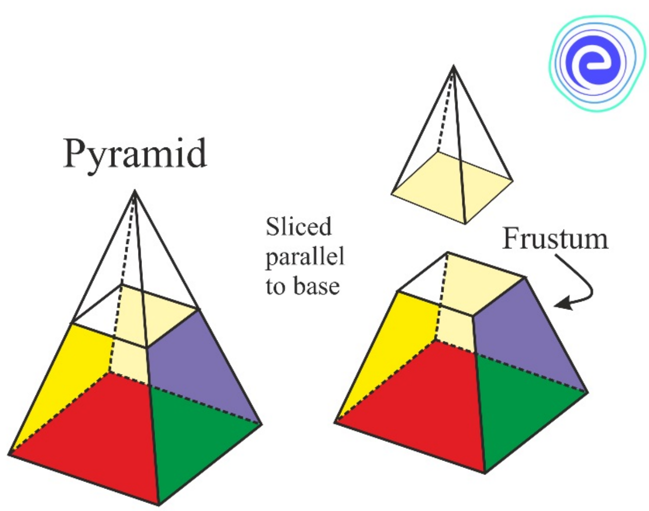


Image from Embibe.com (accessed 8/6/2023).

Information on YBC 4708 (including the image below) is on the Yale Peabody Museum webpage (accessed 8/6/2023). It describes the tablet as a problem text of 60 problems concerning piling bricks. The problems are solved either by using quadratic equations or (in the case of Problems number 49 through 52) using third degree equations. Notice that in the system of equations given above, we can eliminate x and z to get:

$$A = z(x^2+y^2) = (ay+b)((c)^2+y^2) = ac^2y+bc^2+ay^3+by^2 = ay^3+by^2+ac^2y+bc^2.$$

Revised: 8/14/2023