2.5. Babylonia: Algebra

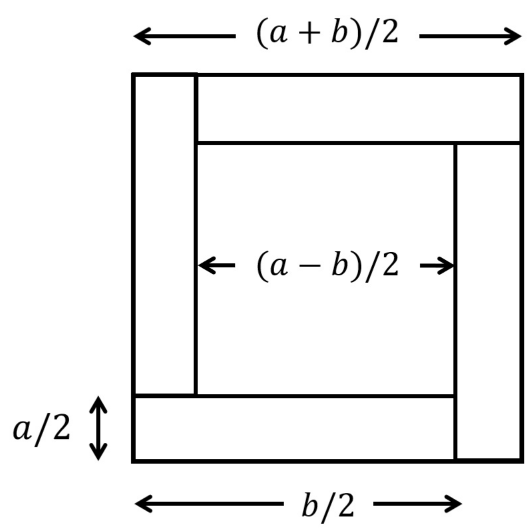
Note. In the previous section, we mentioned that Babylonian algebra was more developed than Babylonian geometry (see Notes 2.4.A). In this section we consider applications of algebra, summations, the use of table texts and polynomial equations, and an estimation of $\sqrt{2}$.

Note 2.5.A. The image at the right is from the Yale Peabody Museum, Babylonian Collection website (accessed 8/5/2023) and gives photos of all sides of the problem text with catalog number YBC 6967 ("YBC" for "Yale Babylon Collection"). The content as described by Peter Rudman in his *How Mathematics Happened: The First 50,000 Years* (Prometheus Books, 2007) as follows (page 231): "The length of a rectangle exceeds its width by 7_{60}. Its area is $1{:}0_{60}$. Find its length and width." Rudman is expressing numbers base 60. Now $7_{60} = 7_{10} = 7$ and $1{:}0_{60} = 60_{10} = 60$. so with the width represented by a and the length represented by b,

the problem translates into our modern notation as: "$b - a = 7$ implies $ab = 60$ $a =?$ and $b =?$" As Rudman explains, the solution is given by the sequence of steps:

	Babylonian Algorithm	Algebraic Generalization
Step 1	$7/2 = 3.5$	$(b-a)/2$
Step 2	$3.5 \times 3.5 = 12.25$	$[(b-a)/2]^2$
Step 3	$12.25 + 60 = 72.25$	$[(b+a)/2]^2 = [(b-a)/2]^2 + ab$
Step 4	$\sqrt{72.25} = 8.5$	$(b+a)/2 = \sqrt{[(b-a)/2]^2 + ab}$
Step 5	$8.5 - 3.5 = 5 = \text{width}$	$a = (b+a)/2 - (b-a)/2$
Step 6	$8.5 + 3.5 = 12 = \text{length}$	$b = (b+a)/2 + (b-a)/2$

In the following figure, Step 1 gives the width of the smaller square, Step 2 gives its area, Step 3 relates the known area of the big square to the (now known) area of the little square plus the areas of the four rectangles, and Step 4 computes the width of the larger square (though taking square roots, in general, is problematic for the Babylonians). Then Steps 5 and 6 give the desired width and length.



Based on Figure 5.4.1 of Rudman.

Note 2.5.B. Tablet AO 6484 of the Louvre Museum is a "Seleucid" tablet of about 300 BCE (see the image below). According to Raymond Archibald's "Babylonian Mathematics," *Isis*, **26**(1), 63–81 (1936) (available on the JSTOR webpage, accessed 8/7/2023), the partial sum of a geometric series is given as (in modern notation, of course):

$$\sum_{i=0}^{9} 2^i = (2^9 - 1) + 2^9 = 1023 = \frac{(2)^{10} - 1}{(2) - 1}.$$

This leads Archibald to speculate that the Babylonians of this time may have known the sum of a geometric progression, $\sum_{i=0}^{n} ar^i = a\frac{r^{n+1} - 1}{r - 1}$; this is given in Euclid's *Elements* in Book IX as Proposition 36.

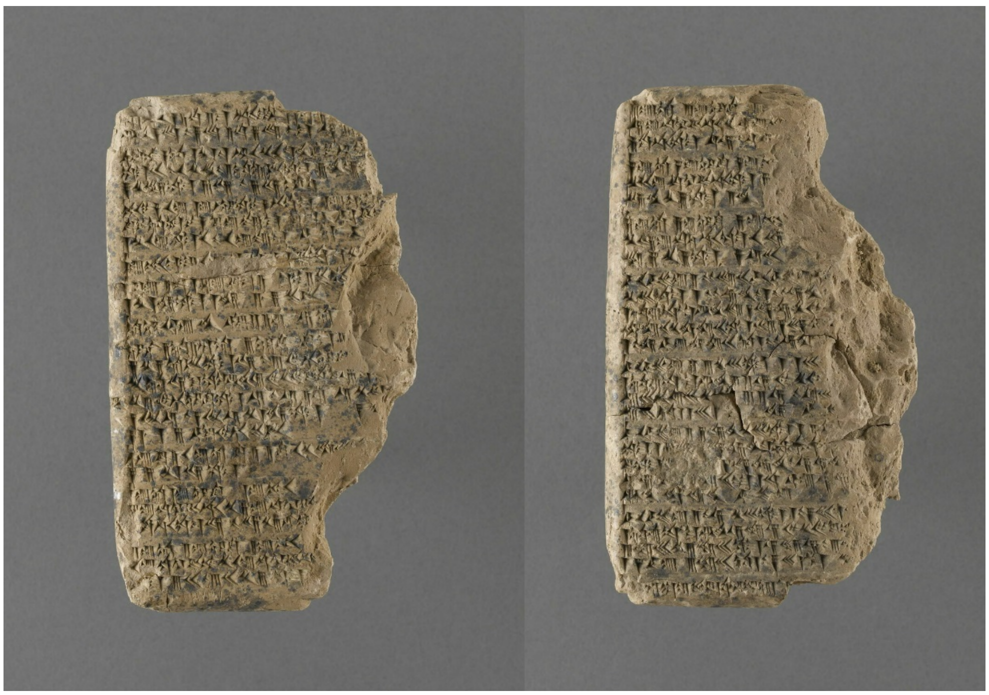


Image of AO 6484 from the Louvre Museum.

On the same tablet, a sum of squares is given:

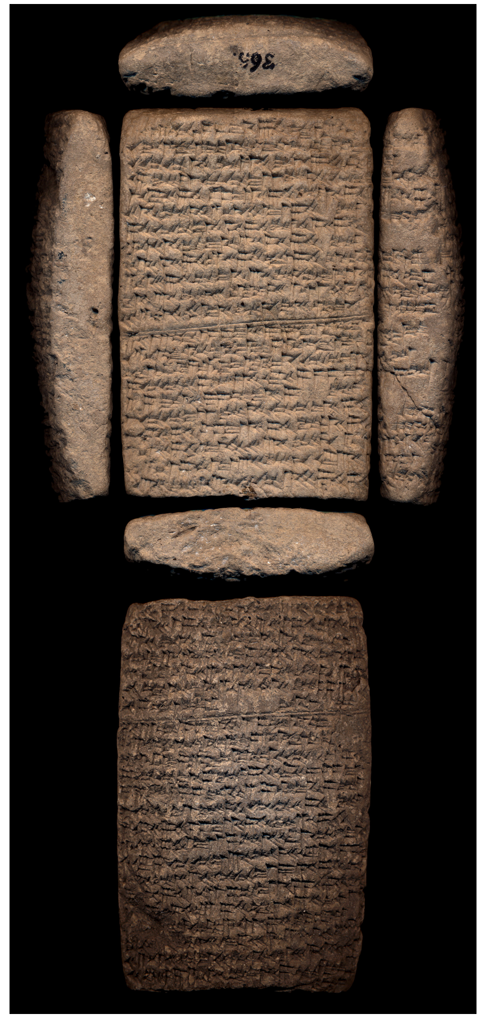
$$\sum_{i=1}^{10} = \left(1\left(\frac{1}{3}\right) + 10\left(\frac{2}{3}\right)\right) 55 = \left(\frac{2(10) + 1}{3}\right) 55 = 385.$$

Archibald then wonders if the Babylonians of 30 BCE knew the formula

$$\sum_{i=1}^{n} i^2 = \frac{2n + 1}{3} \sum_{i=1}^{n} i = \frac{2n + 1}{3} \frac{n(n + 1)}{2} = \frac{n(n + 1)(2n + 1)}{6}.$$

This sum was known to Archimedes (287 BCE–212 BCE), and the sum $\sum_{i=1}^{n} i$ was known to the Pythagoreans (in the form of triangular numbers; see Figure 7 of Section 3.3. Pythagorean Arithmetic). These two examples are also stated by Eves (Eves and Archibald both credit Neugebauer with these observations) on his page 43.

Note 2.5.C. A tablet in the Berlin Museum, with catalog number VAT 8492, lists the values of n^2, n^3, and n^2+n^3 for $n = 1, 2, 3, \ldots, 20, 30, 40, 50$. According to Archibald in his "Babylonian Mathematics," Otto Neugebauer (for a brief biography, see Section 2.2. Babylonia: Sources) surmised that this table was used to solve general cubic equations, $ax^3+bx^2+cx+d = 0$, which had been reduced to a "normal form" $n^3+n^2 = C$. This is to be shown in Problem Study 2.6(d); knowledge of the quadratic formula is necessary to solve Problem Study 2.6(d) but, as we'll see in the next note (Note 2.5.D), the Babylonians knew the quadratic formula. The British Museum has tablets of about 1800 BCE which contain six problems which lead to one-term, three-term, and four-term cubic equations, but only three of these fit Neugebauer's theory (presumably, due to the presence of complex numbers). One of the three-term cases (of which there are two) lead to an equation of the form $(\mu x)^3+(\mu x)^2 = 252$. From the Berlin tablet VAT 8492, one would find that $\mu x = 6$. There is also a large group of table texts which give reciprocals. This type of table allows division problems to be converted to multiplication problems. The source for this note is pages 68 and 69 of Archibald's "Babylonian Mathematics."



Note 2.5.D. An Old Babylonian tablet excavated from Uruk (on the Euphrates River; today it is known as Warka, Iraq), catalogued as Strasbourg tablet number 363, has problems leading to quadratic equations and their solution. The image here is from the Cuneiform Digital Library Initiative website (accessed 8/8/2023). One of these problems is: The sums of the areas of two squares is equal to $A = 2225$. (In modern notation, we let x and y denote the lengths of the sides of the squares, were $x > y$.) The side x of the larger square is equal to a certain quantity $u + d_1$ (where $d_1 = 10$) and the side y of the smaller is equal to $\frac{\alpha}{\beta}u + d_2$ where $\alpha/\beta = 2/3$ and $d_2 = 5$. That is, we have the system of equations (systems of equations are another common topic of the Babylonian tablets):

$$x^2 + y^2 = A, \quad x = u + d_1, \quad y = \frac{\alpha}{\beta} + d_2,$$

where the unknowns are x, y, and u. We eliminate x and y, and perform a change of variables on u. Let $u = W\beta$ so that

$$x^2 = (u + d_1)^2 = (W\beta + d_1)^2 = W^2\beta^2 + 2W\beta d_1 + d_1^2 \text{ and}$$

$$y^2 = (u\alpha/\beta + d_2)^2 = (W\beta\alpha/\beta + d_2)^2 = (W\alpha + d_2)^2 = W\alpha^2 + 2W\alpha d_2 + d_2^2.$$

Then

$$A = x^2 + y^2 = (W^2\beta^2 + 2W\beta d_1 + d_1^2) + (W\alpha^2 + 2W\alpha d_2 + d_2^2)$$

$$= W^2(\alpha^2 + \beta^2) + 2(d_1\beta + d_2\alpha)W + (d_1^2 + d_2^2),$$

or

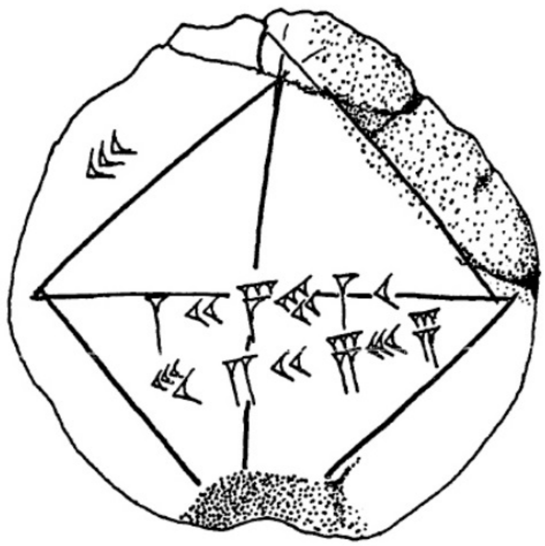
$$W^2 + \frac{2(d_1\beta + d_2\alpha)}{\alpha^2 + \beta^2}W - \frac{A - (d_1^2 + d_2^2)}{\alpha^2 + \beta^2} = 0.$$

We now know that we can take

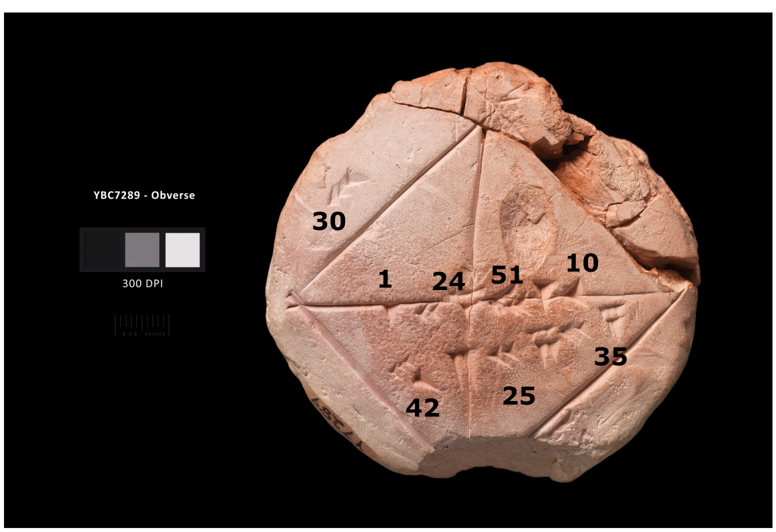
$$\begin{aligned} W &= \frac{1}{2(1)}\left(-\frac{2(d_1\beta + d_2\alpha)}{\alpha^2 + \beta^2} + \sqrt{\frac{4(d_1\beta + d_2\alpha)^2}{(\alpha^2 + \beta^2)^2} - 4(1)\left(\frac{A - (d_1^2 + d_2^2)}{\alpha^2 + \beta^2}\right)}\right) \\ &= \frac{1}{2}\left(-\frac{2(d_1\beta + d_2\alpha)}{\alpha^2 + \beta^2} + \frac{2}{\alpha^2 + \beta^2}\sqrt{(d_1\beta + d_2\alpha)^2 - (A - (d_1^2 + d_2^2))(\alpha^2 + \beta^2)}\right) \\ &= \frac{1}{\alpha^2 + \beta^2}\left(-(d_1\beta + d_2\alpha) + \sqrt{(d_1\beta + d_2\alpha)^2 - (A - (d_1^2 + d_2^2))(\alpha^2 + \beta^2)}\right) \end{aligned}$$

The solution on Strasbourg tablet 363 contains steps equivalent to substitution of the given values into this formula. “There are scores of problems which prove the amazing fact that the Babylonians of 2000 B.C. were familiar with the equivalent of our formula for the solution of a quadratic equation. Until 1929 no one suspected that such a result was known before the time of Heron of Alexandria [circa 10 ce–circa 75 ce] two thousand years later.” This quote and the other observations in this note are based on pages 74 and 75 of Archibald’s “Babylonian Mathematics.”

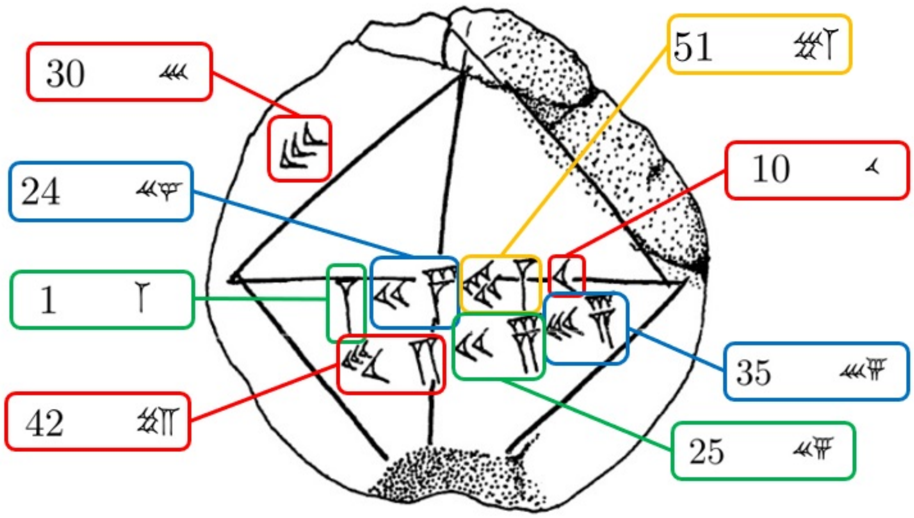
Note 2.5.E. We now consider a surprisingly accurate Babylonian approximation of $\sqrt{2}$. Much of the information in this note appears in Section 1.6. A Remarkable Babylonian Document, which is part of the history component of Introduction to Modern Geometry (MATH 4157/5157). The approximation is performed by finding the length of the diagonal of a square. The computation was performed somewhere between 1900 to 1600 BCE in southern Mesopotamia and is preserved on a clay tablet about 3 inches in diameter (the tablet is denoted YBC 7289). It is now part of the Yale Babylonian Collection. It was donated to the collection by J. P. Morgan (this information is from the Wikipedia webpage on YBC 7289 (accessed 8/8/2023). Details on attempts (with successes and failures) to use images of the tablet in the contemporary classroom environment are described in Janet L. Beery and Frank J. Swetz, "The Best Known Old Babylonian Tablet?," *Convergence* (July 2012) (accessed 8/8/2023).



From the MAA page on "The Best Known Old Babylonian Tablet?"



From Wikipedia page on YBC 7289



A modification of one of the above figures giving the "translation" of the Babylonian numerals into Arabic numerals.

There are three numbers on the tablet. The square is meant to have sides of length 30, as labeled in the upper left. Written on and somewhat under the horizontal diagonal of the square is 1 24 51 10. Converting to Babylonian base 60 (with an implied decimal point after the 1) we have the number :

$$1 \times (60^0) + 24 \times \left(\frac{1}{60}\right) + 51 \times \left(\frac{1}{60^2}\right) + 10 \times \left(\frac{1}{60^3}\right) \approx 1.414212963.$$

Now $\sqrt{2} \approx 1.414213562$, so the horizontal diagonal seems to be labeled with a very good approximation of $\sqrt{2}$ (accurate to 6 decimal places). Under the diagonal is the result of a computation:

$$42 \times (60^0) + 25 \times \left(\frac{1}{60}\right) + 35 \times \left(\frac{1}{60^2}\right) \approx 42.426389.$$

We have $30\sqrt{2} \approx 42.426407$, so it is this number that represents the length of the diagonal, given the length of a side of the square is 30 (accurate to 4 decimal places).

Note. In the previous section we saw that the Babylonians likely had knowledge of the Pythagorean Theorem. In this section we saw that they knew how to sum geometric sequences, they knew the quadratic formula (and considered higher order equations), and they were capable of detailed computations. Eves summarizes this as (see Eves' page 44): "... we conclude that the ancient Babylonians were indefatigable table makers, computers of high skill, and definitely stronger in algebra than geometry. One is certainly struck by the depth and the diversity of the problems that they consider."

Revised: 8/14/2023