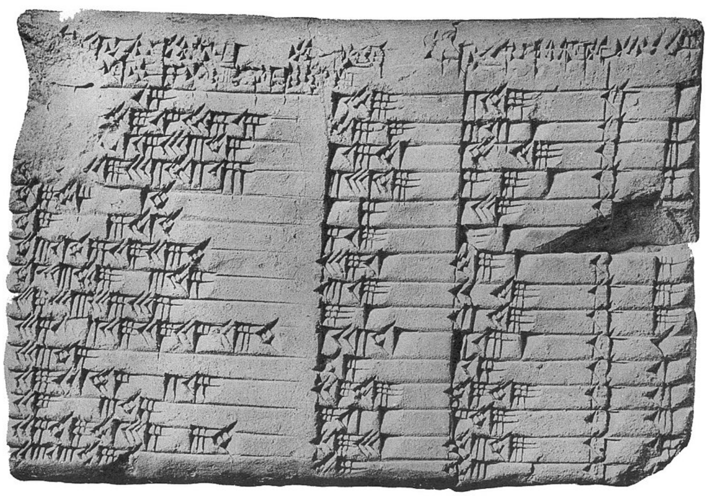
2.6. Babylonia: Plimpton 322

Note. In this section we consider the most famous of the Babylonian mathematical tablets, "Plimpton 322" in the George Plimpton Collection of the Columbia University Libraries. Our primary source for this section is Eves.



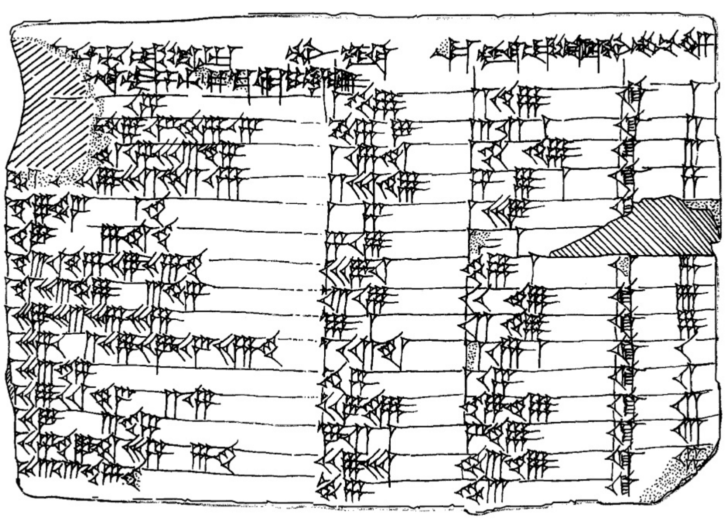


Figure 1. Plimpton 322 (obverse). Drawing by the author.

The images above of Plimpton 322 are from the Wikipedia webpage on Plimpton 322 (upper left), Eleanor Robson's "Words and Pictures: New Light on Plimpton 322," *American Mathematical Monthly*, **109**(2), 105–120 (2002) on the Mathematical Association of America website (lower left), and D. Mansfield and N.J. Wildberger's "Plimpton 322 is Babylonian Exact Sexagesimal Trigonometry," *Historia Mathematica*, **44**, 395–419 (2017) on the Science Direct website (right). Each website was accessed 8/9/2023. Plimpton 322 is from the Old Babylonian period (1900 to 1600 BCE) and was first described in 1945 on pages 38 to 41 by O. Neugebauer and A. Sachs in *Mathematical Cuneiform Texts*, American Oriental Series, vol. 29, eds. O. Neugebauer and A. Sachs, New Haven: American Oriental Society and the American Schools of Oriental Research (1945), a copy of which is online on Google Books (accessed 8/9/2023).

Note. Physically, we see from the images above that there is a deep chip near the middle of the right edge missing. There is a flake missing out of the upper left side. In fact, a missing piece has been broken off from the entire left edge, as can be seen by the abrupt way the table starts on the left side. In addition, crystals of modern glue were found along the broken left side, suggesting that the tablet was broken sometime after its discover and then a repair was attempted. Hopefully, the missing piece is somewhere among the collections of Old Babylonian tablets. The right-most column of numbers, as you likely recognize from our introduction of Babylonian cuneiform numerals in Section 1.4. Simple Grouping Systems, is simply a numbering of the columns 1 through 15, with 5 and 6 missing from the deep chip on the right side, 14 partially visible, and 15 broken off of the lower right corner (Eves page 44).

Note. Reading from right to left, the second column from the right plays no mathematical significance (it appears just to divide the enumeration from the significant numbers). The next two columns (third and fourth from the right) give the hypotenuse (third column from the right) and a leg (fourth column from the right) of integer-sided right triangles. However, there are four mistakes in these columns. The data from Eves' Figure 4 (with the absent length of the other leg of the right triangle (in blue, and Eves' corrections in red) are as follows:

u	v	Absent Leg	Leg	Hypotenuse	#
12	5	120	119	169	1
64	27	3456	3367	(111521) 4825	2
75	32	4800	4601	6649	3
125	54	13500	12709	18541	4
9	4	72	65	97	5
20	9	360	319	481	6
54	25	2700	2291	3541	7
32	15	960	799	1249	8
25	12	600	(541) 481	769	9
81	40	6480	4961	8161	10
2	1	60	45	75	*11
48	25	2400	1679	2929	12
15	8	240	(25921) 161	289	13
50	27	2700	1771	3229	14
7	2	90	56	(53) 106	*15

Parameters u and v (not part of Plimpton 322) given in the first two columns of this table, and the asterisks* will be explained below. Eves' offers explanations of the last three mistakes on the tablet. In line 9, the correct value in base 60 is $8,1$ (that is, $8 \times 60 + 1$) and the incorrect value 541 is $9,1$ (that is, $9 \times 60 + 1$) in sexagesimal. So an extra character is included in the 60's place. In line 13, the correct value 161 when squared gives the incorrect value 25921 (though 25921 is not the *correct* value here, it is a *relevant* value here since we are considering right triangles). The line 15, incorrect value 53 is half of the correct value 106. In other words, each of these is a simple typographical (or "scripto-graphical") error! The resolution of the mistake in line 2 "has received an involved explanation," for which Eves references Jöran Friberg's "Methods and traditions of Babylonian mathematics: Plimpton 322, Pythagorean Triples, and the Babylonian Triangle Parameter Equations," *Historia Mathematica*, **8**(3), 277–318 (1981); this is available online on Science Direct (accessed 8/9/2023).

Note. You may have been introduced to Pythagorean triples in Elementary Number Theory (MATH 3120); see my online notes for Elementary Number Theory on Section 16. Pythagorean Triangles. We will also see them again in this class in Section 3.4. Pythagorean Theorem and Pythagorean Triples. A *Pythagorean triple* is three positive integers a, b, c such that $a^2 + b^2 = c^2$. One can see the obvious connection with the Pythagorean Theorem (we saw evidence that the Babylonians knew of the Pythagorean Theorem in Section 2.4. Babylonia: Geometry). In fact, Plimpton 322 is additional evidence that the Babylonians had knowledge of and an interest in the Pythagorean Theorem. Easy examples of Pythagorean triples are $3, 4, 5$, since $3^2+4^2 = 5^2$, and $5, 12, 13$, since $5^2+12^2 = 13^2$. A Pythagorean triple is *primitive* if there is no integer (greater than 1) that is a factor of all three members of the triple. An example of a non-primitive Pythagorean triple is $45, 60, 75$; notice that $45^2 + 60^2 = 75^2$ but 15 is a factor of each. In fact, line 11 of Plimpton 322 gives the Pythagorean triple $45, 60, 75$ (though only 45 and 75 are on the tablet). This is the reason that lines 11 and 15 are marked with asterisks*.

Note. The generation of primitive Pythagorean triples is totally classified in Elementary Number Theory (MATH 3120). The following theorem is from Section 16. Pythagorean Triangles of that class:

Theorem 16.1. All solutions $x = a$, $y = b$, $z = c$ to $x^2 + y^2 = z^2$, where a, b, c are positive integers and have no common factor and a is even, are given by $a = 2uv$, $b = u^2 - v^2$, $c = u^2 + v^2$, where u and v are any relatively prime integers, not both odd, and $u > v$.

In Lemma 16.1 of the same section of notes, it is shown that exactly one of a and b must be even, so Theorem 16.1 is in fact a classification of primitive Pythagorean triples. This explains the parameters u and v in the table given above. . . with two arguable exceptions. In line 11, as observed above, the given Pythagorean triple is not primitive. The given triple $45, 60, 75$ is a multiple of the primitive Pythagorean triple $3, 4, 5$ (namely, the multiple 15), and the primitive triple is based on $u = 2$ and $v = 1$. In line 15, the Pythagorean triple $56, 90, 106$ is not primitive since it is a multiple of the primitive Pythagorean triple $28, 45, 53$ (the multiple is 2), and the primitive triple is based on $u = 7$ and $v = 2$. "The evidence seems good that the Babylonians of this remote period were acquainted with the general parametric representation of primitive Pythagorean triples as given above [in Theorem 16.1]" (Eves, page 46).

Note. A natural number is (sexagesimally) *regular* if its reciprocal has a finite (sexagesimal) expansion. In Problem Study 2.1(a) is to be shown that a necessary and sufficient condition for n to be regular is that $n = 2^a 3^b 5^c$, where a, b, c are nonnegative integers. Notice that all the values of u and v given in the table above are regular numbers, except for $u = 7$ in line 15. This exception can be resolved by taking $u = 9$ and $v = 5$ (both regular), for which $a = 2uv = 90$, $b = u^2 - v^2 = 56$, and $c = u^2 + v^2 = 106$. Since both $u = 9$ and $v = 5$ are odd then Theorem 16.1 does not apply, so that the Pythagorean triple $90, 56, 106$ is not primitive, as we commented above. "It appears that the table on the tablet was constructed by deliberately choosing small regular numbers for the parameters u and v" (Eves, page 47). For u and v regular, $a = 2uv$ is also regular. This means that reciprocals of the values of a have finite sexagesimal expansions so that division by a can be converted to multiplication by $1/a$. This comes into use when we consider the left-most column (which is partially destroyed). The pattern that has been established from the partial column is that it contains the values of $(c/a)^2 = [(u^2 + v^2)/(2uv)]^2$ (*this* is why regular u and v are used to generate the values on Plimpton 322). This means that this column represents values of the secant function squared evaluated at the angle B opposite side $b = u^2 - v^2$. In fact, the particular choices of triangles yield almost evenly spaced decreases (by an amount of $1/60$) of $\sec B$ as angle B decreases from 45° to 31°. That is, triangles with integer-length sides have been used to produce a secant table for angles between 45° to 31° where there is a regular jump in the function, rather than in the corresponding angle (this is unintuitive to us and we would expect that evenly spaced inputs would be used to give the function outputs; the Babylonians, however, did not have a function but they could determine function values if they could find appropriately sized right triangles). It seems likely that there were similar tables for angles between 1° and 15° and between 16° and 30° (Eves, page 47). In D. Mansfield and N.J. Wildberger's "Plimpton 322 is Babylonian Exact Sexagesimal Trigonometry," *Historia Mathematica*, **44**, 395–419 (2017) (online on the Science Direct website; accessed 8/9/2023), the techniques used to generate Plimpton 322 are further discussed, and extensions of it to other values are given. In their Table 9 they give the 15 values on Plimpton 322 for angled between 45° and 31°, and then add an additional 23 Pythagorean triples to get values of $\sec^2 B$ and $\sec B$ for angles less than 31° (this gives speculated versions of the unknown tablets predicted by Eves). Their Table 9 gives values in sexagesimals, and in their Table 13 they give the same information base 10. Again using Pythagorean triples, they extend the chart down to a secant value of 1.00083333 (corresponding to an angle $B \approx 2°20'$).

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