2.8. Egypt: Arithmetic and Algebra

Note. In this section, we consider Egyptian techniques of computation and problems from the Moscow and Rhind Mathematical Papyri.

Note 2.8.A. Egyptian multiplication and division is usually performed by a succession of doubling operations. This is based on the fact that any natural number can be written as a sum of powers of 2 (this is the basis for the use of binary representations of numbers). For example, to multiply 26 and 33, we can represent 26 as a sum of powers of 2: $26 = 2 + 2^3 + 2^4 = 2 + 8 + 16$. Then we have $26 \times 33 = (2+8+16) \times 33 = 2 \times 33 + 8 \times 33 + 16 \times 33$. We can then generate the products by repeatedly doubling numbers, starting with 33:

$$\begin{array}{rcrcr} 1 & \times & 33 & = & 33 \\ {}^*2 & \times & 33 & = & 66 \\ 4 & \times & 33 & = & 132 \\ {}^*8 & \times & 33 & = & 264 \\ {}^*16 & \times & 33 & = & 528 \end{array}$$

The entries marked with an asterisk are the ones we need to determine the product, so that $26 \times 33 = 66 + 264 + 528 = 858$. To divide, the Egyptians perform a similar procedure. For example, to divide 753 by 26 we consider powers of 2 times 26:

$$\begin{array}{rcrcr} 1 & \times & 26 & = & 26 \\ 2 & \times & 26 & = & 52 \\ {}^*4 & \times & 26 & = & 104 \\ {}^*8 & \times & 26 & = & 208 \\ {}^*16 & \times & 26 & = & 416 \end{array}$$

Now we subtract from 753 the largest power of 2 multiple of 26 that is not greater than 753, and get a remainder. We then apply this same technique to the remainder, and iterate this process until we have a remainder less than 26:

$$753 = 416 + 337 = 416 + 208 + 129 = 416 + 208 + 104 + 25.$$

So the quotient is determined from the powers of 2 with an asterisk, $16+8+4 = 28$, and the remainder is 25. Eves praises this technique as (see his page 53): "This Egyptian process of multiplication and division not only eliminates the necessity of learning a multiplication table, but is so convenient on the abacus that it persisted as long as that instrument was in use, and even for some time beyond."

Note 2.8.B. Translation of Problem 6 of the Moscow Mathematical Papyrus is: "Given a rectangular enclosure of 12 units area and the ratio of the sides as 1 : 3/4, find the lengths of the sides." With x as the length of the longer side of the rectangle, we have $3x/4$ as the length of the shorter side. Then $3x^2/4 = 12$ or $x^2 = 16$ or $x = 4$ and $x = 3$. The following image is Problem 6 as it appears on the Rhind Papyrus.

This translates as:

Line 1. Example of Calculating a rectangle.

Line 2. If someone says to you: A rectangle is 12 setjat [in area] has a breadth 1/2 1/4 [i.e., summing, 3/4] of its length. [Calculate it.]

Line 3. Calculate 1/2 1/4 to get 1. The result is 1 1/3.

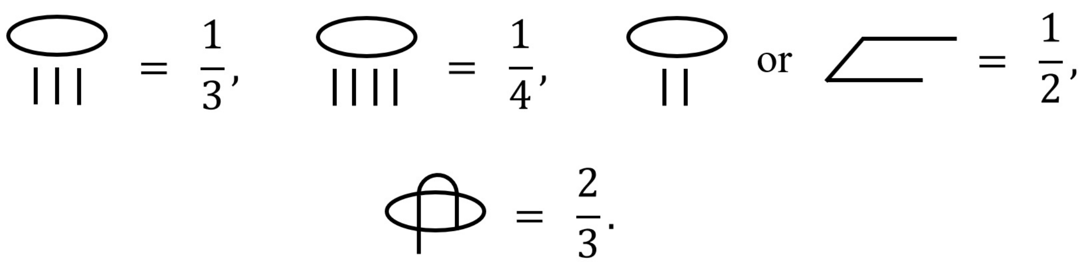
Line 4. Take this 12 setjat 1 1/3 times. The result is 16.

Line 5. Calculate its square root. The result is 4 for its length [and] 1/2 1/4 of it is 3 for the breadth.

Line 6. [This includes an illustration showing a rectangle with area 12 written in the interior, 4 at the top (there is a typo in the image which has a 3, "| | |," on top of the rectangle; the flawed image is from M. Clagett *Ancient Egyptian Science: Ancient Egyptian Mathematics* (1999)), and 3 at its side.]

This problem shows that the Egyptians were familiar with square roots. In fact, the symbol  in the fifth line is the hieroglyph for taking a square root. The reference for this note (and the source of the image) is the Saint Louis University webpage on Math & the Art of MC Escher (accessed 8/12/2023), which references R.C. Archibald *Mathematics before the Greek Science, New Series*, **71**(1831), 109–121 (1930) in the translated statement and solution of Problem 6.

Note. We have encountered the Egyptian idea of unit fractions before, here a fraction is expressed as the sum of reciprocals of integers and the one special fraction 2/3 (see Note 2.7.D on the Rhind Papyrus in Section 2.7. Egypt: Sources and Dates). Unit fractions are given as hieroglyphs by placing an elliptical symbol over the denominator. The hieroglyphs for 1/3, 1/4, 1/2, and the special fraction 2/3 are as follows (based on Eves' page 53):



In hieroglyphs, 3/4 would be represented (as the translations given above of Lines 2, 3, and 5 of Problem 6 of the Moscow Mathematical Papyrus) by placing the hieroglyphs for 1/4 and 1/2 side-by-side. This can be seen in the hieroglyphic version of Problem 6 as given in Note 2.8.B above. The hieroglyphs  appear in Lines 2, 3, and 5 (with the hieroglyph for 1/2 with a rounder top and reversed from the version given by Eves).

Note 2.8.C. We mentioned "pefsu problems," related to the strength of beer, in the Moscow Mathematical Papyrus in Section 2.7. Egypt: Sources and Dates (see Note 2.7.C). An example of this is given on the Wikipedia webpage for the Moscow Mathematical Papyrus. A *pefsu* measures the strength of the beer made from a hekat of grain

$$\text{pefsu} = \frac{\text{number loaves of bread (or jugs of beer)}}{\text{number of heqats of grain}}$$

A higher pefsu number means weaker bread or beer. The Wikipdia page includes a translation of Problem 8 (based on the numbering scheme of V. V. Struve) and gives as the reference of this translation Marshall Clagett's *Ancient Egyptian Science: A Source Book. Volume 3: Ancient Egyptian Mathematics*, Memoirs of the American

Philosophical Society 232 (Philadelphia: American Philosophical Society, 1999). The problem reads:

(1) Example of calculating 100 loaves of bread of pefsu 20

(2) If someone says to you: "You have 100 loaves of bread of pefsu 20

(3) to be exchanged for beer of pefsu 4

(4) like 1/2 1/4 malt-date beer"

(5) First calculate the grain required for the 100 loaves of the bread of pefsu 20

(6) The result is 5 heqat. Then reckon what you need for a des-jug of beer like the beer called 1/2 1/4 malt-date beer

(7) The result is 1/2 of the heqat measure needed for des-jug of beer made from Upper-Egyptian grain.

(8) Calculate 1/2 of 5 heqat, the result will be 2 1/2

(9) Take this 2 1/2 four times

(10) The result is 10. Then you say to him:

(11) "Behold! The beer quantity is found to be correct."

To interpret this, we first have to make sense out of the "1/2 1/4 malt-date beer" statement. This indicates that the beer is made of 1/2 malt (malt is grain that has started germinating by soaking in water, but then germination is stopped by drying; this generates the sugar that is fermented in making the beer), 1/4 dates, and presumably 1/4 something else. The "1/2 malt" seems to indicate that only half of the amount of grain is to be used (this is the computation performed in Step (8)). Steps (5) and (6) are clear, and follow from the given formula as $20 = 100/5$; so applying the given equation to the bread we have 5 heqat of grain. For the beer, we cut the amount of grain in half to get 2.5 heqat of grain (Step (8)). Applying the given equation to beer of pefsu 4 based on 2.5 heqat of grain gives $4 = 10/2.5$, so that the number of jugs of beer is 10.

Note 2.8.D. In Section 2.7. Egypt: Sources and Dates that the first part of the Rhind Papyrus contains a small table of the numbers 1 through 9 divided by 10 expressed as sums of unit fractions (that is, sums of reciprocals of integers); see Note 2.7.D. The first six problems of the Rhind Papyrus involve applications of this table. The problems concern dividing 1, 2, 6, 7, 8, and 9 loaves of bread among 10 men. The problems are solved using the table just mentioned, with respective solutions: $1/10$, $2/20 = 1/5$, $6/10 = 1/2 + 1/10$, $7/10 = 2/3 + 1/30$, $8/10 = 2/3 + 1/10 + 1/30$), and $9/10 = 2/3 + 1/5 + 1/30$. Eves says on his page 53 in the next section that: "The Egyptians... representing all fractions, except 2/3, as the sum of so-called unit fractions, or fractions with unit numerators." Problem Study 2.9 gives other problems related to unit fractions. The statements of problems 21 through 38 as given in the papyrus can appear complicated (especially in prose used), each problem ultimately reduces to a simple linear equation. As examples (with solutions expressed using unit fractions), we have

Problem 24. $x + \frac{1}{7}x = 19$. The solution is $x = 16 + \frac{1}{2} + \frac{1}{8}$.

Problem 31. $x + \frac{2}{3}x + \frac{1}{2}x + \frac{1}{7}x = 33$. The solutions is

$$x = 14 + \frac{1}{4} + \frac{1}{56} + \frac{1}{97} + \frac{1}{194} + \frac{1}{388} + \frac{1}{679} + \frac{1}{776}.$$

Problem 40, in modern terms, is the following:

> 100 loaves of bread are to be divided among five men. The men's five shares of bread are to be in arithmetic progression, so that consecutive shares always differ by a fixed difference, or Δ. Furthermore, the sum of the three largest shares is to be equal to seven times the sum of the two smallest shares. Find Δ and write it as an Egyptian fraction.

This problem can be set up in terms of a system of equations (making it approach close to a topic in linear algebra!). Its solution is to be given in Problem Study 2.8.A. This is the last problem in Part I (algebra and arithmetic) of the Rhind Papyrus. Other examples of Egyptian algebra are given in Problem Study 2.12. The source for this note is the Wikipedia webpage on the Rhind Mathematical Papyrus (accessed 8/12/2023).

Note 2.8.E. Many Moscow and Rhind Papyrus problems reduce down to simple linear equations. They are often solved, not by the algebraic manipulations we are used to, but by the *rule of false position* (as it was later known in Europe). For example, to solve $x + x/7 = 24$, a convenient guess is made, say $x = 7$. Now with $x = 7$ we have $x + x/7 = (7) + (7)/7 = 8$. Since this is 1/3 of the desired right-hand-side (and there are only multiples of the unknown on the left-hand-side), we can modify the guess from $x = 7$ to $x = 7 \times 3 = 21$. Of course the rule of false position is only applicable to linear equations of a certain form (with the unknown on one side and a constant on the other).

Note 2.8.F. The Kahun Papyri make up the largest collection of papyri ever found. Most date from about 1825 BCE. They include works on adminstrative matters, medical topics (including verterinarian topics), and mathematics. The latter of these make up the Lahun Mathematical Papyri (sometimes called the Kahun Mathematical Papyri, as Eves does). It consists of six fragments. Parts of the Lahun Papyri can be viewed on the University College of London Digital Egypt webpage (accessed 8/13/2023). The papyrus has a table of Egyptian fractions representing numbers of the form $2/n$ (in Lahun IV.2), as does the Rhind Papyrus. It considers numbers in arithmetical progression (as does the Rhind Papyrus) in Lahun IV.3. In Lahun LV.4 there are baku problems similar to those of the Moscow and the Rhind Mathematical Papyri. Eves describes the following problem from the Lahun Papyrus (Eves' page 55): "A given surface of one hundred units of area shall be represented as the sum of two squares whose sides are to each other as $1 : 3/4$." The problem is the find the lengths of the sides of the two squares. If we denote the lengths of these sides as x and y, then we have $x^2 + y^2 = 100$ and $x = 3y/4$. Using the rule of false position of Note 2.8.E, we try the convenient value $y = 4$ from which we get $x = 3$ and $x^2 + y^2 = 25$. We have only squared unknowns on the left-hand-side and only a constant on the right-hand-side. Since we want 100 on the right-hand-side, we need to increase 25 by a factor of 4. This can be done by increasing both x and y by a factor of 2 (which gets squared to give the desired factor of 4). So we take as a solution $x = 6$ and $y = 8$. Of course $(6)^2 + (8)^2 = 100$, as desired. In fact, this is related to the primitve Pythagorean triple $3, 4, 5$ (see Section 2.6. Babylonia: Plimpton 322). In addition to Eves, the source for this note is the Wikipedia webpage on the Lahun Mathematical Papyrus (accessed 8/13/2023).

Note. At the end of this section, Eves addresses the use of symbols in Egyptian algebra: "In the Rhind papyrus, we find symbols for *plus* and *minus*. The first of these symbols represents a pair of legs walking from left to right and the other a pair of legs walking from right to left" (page 55). He also claims that symbols were used for *equals* and *unknown*. The only other readily available references which address this are Joseph Mazur's *Enlightening Symbols: A Short History of Mathematical Notation and Its Hidden Powers* (Princeton University Press, 2014), which on page 101 states: "Ahmes [the scribe of the Rhind Papyrus] used a hieratic sign of a papyrus-roll to mean the unknown," and Florian Cajori's *A History of Mathematical Notations*, Volume 1: Notations in Elementary Mathematics (Open Court Publishing, 1928; reprinted by Dover Publications, 1993) which observes on page 18 (paragraph 26) that the Egyptians had a square root symbol (as described above in Note 2.8.B) and that for a quotient, the symbol $\begin{smallmatrix}\infty\\\infty\\\infty\end{smallmatrix}$ was used.

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