2.9. Egypt: Geometry

Note. In this section, we consider some geometry problems from the Moscow and Rhind Mathematical Papyri.

Note 2.9.A. Based on V. V. Struve's enumeration of the problems in the Moscow Mathematical Papyrus, Problem 10 involves the surface area of a "basket." The basket has roundness, but there is not universal agreement as to whether the problem involves a hemisphere or a semi-cylinder. We give the statement given on the Wikipedia webpage on the Moscow Mathematical Papyrus (accessed 8/13/2023):

> "Example of calculating a basket. You are given a basket with a mouth of $4\,1/2$. What is its surface? Take $1/9$ of 9 (since) the basket is half an egg-shell. You get 1. Calculate the remainder which is 8. Calculate $1/9$ of 8. You get $2/3 + 1/6 + 1/18$. Find the remainder of this 8 after subtracting $2/3 + 1/6 + 1/18$. You get $7 + 1/9$. Multiply $7 + 1/9$ by $4 + 1/2$. You get 32. Behold this is its area. You have found it correctly."

The computation should be giving the surface area of a hemisphere of diameter $d = 4\,1/2$. The value 9 represents twice the diameter, $2d$. This is multiplied by $1/9$ and the result subtracted from twice the diameter; this gives $(2d)(8/9)$, which for $d = 4\,1/2$ is 8. This is then multiplied by $1/9$ to give $(2d)(8/81)$, which for $d = 4\,1/2$ is $8/9 = 2/3 + 1/6 + 1/18$. The last quantity is then subtracted from $(2d)(8/9)$ to give $(2d)(8/9) - (2d)(8/81) = (2d)(64/81) = 128d/81$, which for $d = 4\,1/2$ is $7\,1/9$. Finally, this quantity is multiplied by d to give the area of the hemisphere as $128d^2/81$, which is 32. Since the surface area of a sphere of radius

r is $A = 4\pi r^2 = \pi(2r)^2 = \pi d$, then the surface areas of a hemisphere is $\pi d^2/2$. If we set the Moscow Papyrus approximation of the area equal to the actual area, we get $128d^2/81 = \pi d^2/2$, so this results in an approximation $\pi \approx 256/81 \approx 3.16049$. In fact, this is the same approximation of π that appears in Problems 41 and 48 of the Rhind Papyrus (see Note 2.7.D of Section 2.7. Egypt: Sources and Dates).

Note 2.9.B. Problem 14 of the Moscow Papyrus involves the volume of a frustum of a pyramid (see Note 2.4.D of Section 2.4. Babylonia: Geometry). The statement as given on the Wikipedia webpage on the Moscow Mathematical Papyrus (accessed 8/13/2023) is:

> "If you are told: a truncated pyramid of 6 for the vertical height by 4 on the base by 2 on the top: You are to square the 4; result 16. You are to double 4; result 8. You are to square this 2; result 4. You are to add the 16 and the 8 and the 4; result 28. You are to take 1/3 of 6; result 2. You are to take 28 twice; result 56. See, it is of 56. You will find [it] right."

(Notice the very "cook book" approach to the solution.) We take a is the length of the side of the larger square, b as the length of the side of the smaller square, and h as the height. So we are given $a = 4$, $b = 2$, and $h = 6$. We are to square the 4 giving 16 (or, in general, a^2). We are to double the 4 giving 8 (or ab; the "doubling" is done because $b = 2$, so we are not getting $2a$ here). We are to square the 2 giving 4 (or b^2). We add the 16 and the 8 and the 4, giving 28 (or, $a^2 + ab + b^2$). We take 1/3 of 6 giving 2 (or $h/3$). We take 28 twice giving 56 (or $h(a^2 + ab + b^2)/3$; again, the "twice" because $h/3 = 2$). Therefore, Problem 14 gives the volume of a frustum with the given dimensions as $V = h(a^2 + ab + b^2)/3$, the correct formula. In fact, this is Problem Study 2.14(a).

Note 2.9.C. As seen in Section 2.7. Egypt: Sources and Dates (see Note 2.7.D), Part II of the Rhind Mathematical Papyrus, which contains Problems 41 to 60, is on geometry. For statements of problems from the Rhind Papyrus in this section, we refer to the book: Arnold B. Chace (with assistance of Henry P Manning), *The Rhind Mathematical Papyrus*, British Museum 10057 and 10058, Volume I, Free Translation and Commentary (Mathematical Association of America, 1927). This is available online on the Wikipedia.org (accessed 8/14/2023). Problem 41 is stated on page 86 of Chace's book as:

> *Problem 41*
>
> *Find the volume of a cylindrical granary of diameter 9 and height 10.*
>
> Take away 1/9 of 9, namely 1; the remainder is 8. Multiply 8 times 8; it makes 64. Multiply 64 times 10; it makes 640 cubed cubits. Add 1/2 of it to it; it makes 960, its contents in *khar*. Take 1/20 of 960, namely 48. 4800 *hekat* of grain will go into it.

We are looking for the volume of a cylinder of diameter $d = 9$ and height $h = 10$. We'll find the solution similar to that given in Problem 10 of the Moscow Papyrus (see Note 2.9.A above). We go through each step and express it in general. Take away 1/9 of 9, namely 1; the remainder is 8 (in general, $d(8/9) = 8d/9$). Multiply 8 times 8; it makes 64 (or $(8d/9)^2 = 64d^2/81$). Multiply 64 times 10; it makes 640 cubed cubits (or $h(64d^2/81 = 64d^2h/81)$. This is the approximation of the volume. Since the volume of a cylinder of radius r is $V = \pi r^2 h$, we have $V =$

$\pi r^2 h \approx 64d^2h/81 = 64(2r)^2h/81 = 256r^2h/81$, or $\pi \approx 256/81 \approx 3.16049$ (the same approximation that we had in Problem 10 of the Moscow Papyrus). The remaining steps are unit conversions. Multiplying cubic cubits by 3/2 ("Add 1/2 of it to it") converts the volume to *khar*. Dividing the number of *khar* by 20 ("Take 1/20 of [it]") converts the volume of *khar* into hundreds of *hekats* of grain.

Note 2.9.D. With an approximate value of π as 256/81, other area problems can be considered. Problem 48 of the Rhind Papyrus, as stated by Chace on his page 91 is:

Problem 48

Compare the area of a circle and of its circumscribing square.

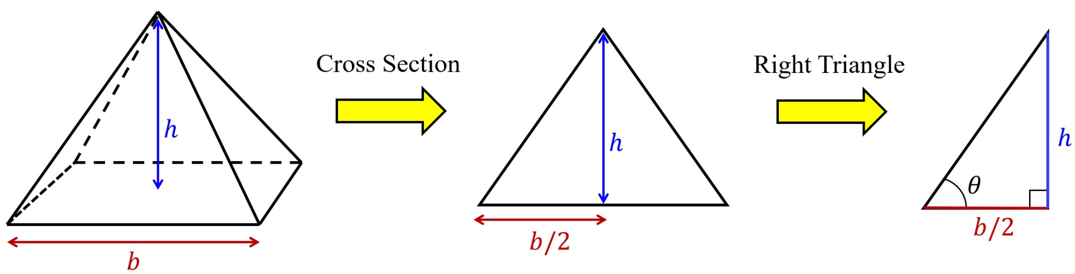
The problem symbolically is to approximate $(\pi r^2)/(2r)^2 = (\pi d^2/4)/d^2 = \pi/4$ The approximation is then $(256/81)/4 = 64/81$. The solution as given in the Rhind Papyrus involves specific computations using the value $d = 9$ and the use of powers of 2 to perform the multiplication, as illustrated in Note 2.8.A of Section 2.8. Egypt: Arithmetic and Algebra.

Note 2.9.E. In Section 2.7. Egypt: Sources and Dates we mentioned the *seked* of a pyramid (see Note 2.7.D). This idea is related to the slope of the side of a pyramid, but it isn't simply the slope. Problem 56 of the Rhind Papyrus concerns the *seked* of a pyramid of certain dimensions measured in cubits (a cubit is roughly the length of a forearm and is taken to be about 18 inches). As we'll explain, the *seked* can be measured in cubits so that it is somewhat different from slope (which is unitless). The problem as stated by Chace on his pages 96 and 97 is:

Problem 56

> *If a pyramid is 250 cubits high and the side of its base 360 cubits long, what is its* seked?
>
> Take 1/2 of 360; it makes 180. Multiply 250 so as to get 180; it makes 1/2 1/5 1/50 [these are to be summed] of a cubit. A cubit is 7 palms. Multiply 7 by 1/2 1/5 1/50. The *seked* is 5 1/25 palms.

In this problem, the *seked* is interpreted as the cotangent of angle θ as the base of a right triangle with base half the length of the side of the base b of the pyramid and height equal to the height h of the pyramid. Notice $\cot\theta = (b/2)/h$.



In this problem we have $h = 250$ cubits and $b = 360$ cubits. We take 1/2 of 360 to get 180 (or, in general, $b/2$). To "[m]ultiply 250 so as to get 180" is to take the quotient $180/250 = 18/25 = 1/2 + 1/5 + 1/50$ (or $(b/2)/h$); this is where the cotangent of θ is introduced. Notice that $\cot\theta$ is just the reciprocal of the slope of the side of the pyramid ($\cot\theta$ is the "run" divided by the "rise"). Now by definition, the *seked* is a measure of the "run" of the side of the pyramid per unit (i.e., per cubit) "rise" of the side. So the *seked* is $\cot\theta = (b/2)/h$ with units of cubits. Since a cubit is 7 palms, we convert the *seked* in cubits to units of plams by multiplying by 7 to get $7 \times (1/2 + 1/5 + 1/50) = 5 + 1/25$. In general, if a and b are measured in cubits, then the *seked* is $7\cot\theta = 7(b/2)/h = 7b/(2h)$ palms.