3.3. Pythagorean Arithmetic

Note. In this section we consider several topics from number theory which are commonly attributed to Pythagoras or to his school. We consider divisors and proper divisors of positive integers (we denote the set of positive integers as $\mathbb{N}$, commonly called the *natural numbers*; notice that, by convention, we do not include 0 in $\mathbb{N}$... this is not a universal convention). Based on these, we define amicable pairs of numbers, perfect numbers, deficient numbers, and abundant numbers. We introduce the function $\sigma(N)$ which is the sum of the divisors of N. We define figurate numbers (triangular numbers, square numbers, pentagonal numbers, etc.) and give formulae for some of these. For each topic, we give some history and quote references that tie the ideas to Pythagoras.

Note. The ancient Greeks labeled the study of abstract relationships between numbers (what we might think of as the theoretical study of numbers) as *arithmetic*, and practical computations with numbers as *logistic*. Today, the term "arithmetic" usually refers to the computational side (at an elementary level), and *number theory* is the contemporary term for the theoretical study of numbers. At ETSU, you likely first encounter number theory in Mathematical Reasoning (MATH 3000). See my online notes for Mathematical Reasoning, and notice "Chapter 6. Number Theory." We have a junior level class devoted to number theory (though it is taught only sporadically), Elementary Number Theory (MATH 3120). I have online notes for Elementary Number Theory which include supplements Fermat's Last Theorem—History, The Prime Number Theorem—History, and The Riemann Hypothesis—History.

Note 3.3.A. Iambilichus (circa 250–circa 325), writing roughly 750 years after Pythagoras lived, credits Pythagoras of the discovery of *amicable number* (or "friendly numbers"). These are introduced in Elementary Number Theory (MATH 3120) Section 8. Perfect Numbers (see the later part of the section). We give some formal definitions first, and then illustrate them.

Definition. A *divisor* of a positive integer $N \in \mathbb{N}$ is a positive integer that evenly divides. We denote the fact that positive integer M divides N as $M \mid N$. A *proper divisor* (or in antiquated terminology, "aliquot part") of positive integer N is a (positive) divisor of N other than N itself (notice that 1 is a proper divisor of all N). Two numbers are *amicable numbers* (or form an "amicable pair") if each is the sum of the proper divisors of the other.

Note. An example of amicable numbers is the pair 220 and 284. The proper divisors of 220 are 1, 2, 4, 5, 10, 11, 20, 22, 44, 55, 110, which sum to 284. The proper divisors of 284 are 1, 2, 4, 71, 142, which sum to 220. As commonly happens with ideas associated with Pythagoras, this pair of numbers took on a mystical meaning. "The numbers came to play an important role in magic, sorcery, astrology, and the casting of horoscopes" [Eves, page 76]. This pair keeps its unique status as the only known pair of amicable numbers (apparently) until Pierre de Fermat in 1626 announced the pair 17,296 and 18,416 as another pair. Some additional historical details are given on the Wikipedia page on Amicable Numbers, as follows (accessed 2/16/2023):

> "A general formula by which some [amicable numbers] could be derived was invented circa 850 by the Iraqi mathematician Thābit ibn Qurra (826–901). Other Arab mathematicians who studied amicable numbers are al-Majriti (died 1007), al-Baghdadi (980–1037), and al-Fārisī (1260–1320). The Iranian mathematician Muhammad Baqir Yazdi (16th century) discovered the pair (9363584, 9437056), though this has often been attributed to Descartes. ... Thabit ibn Qurra's formula was rediscovered by Fermat (1601–1665) and Descartes (1596–1650), to whom it is sometimes ascribed, and extended by Euler (1707–1783). ... Fermat and Descartes also rediscovered pairs of amicable numbers known to Arab mathematicians."

The Wikipedia page lists the smallest 10 amicable pairs:

#	1	2	3	4	5	6	7	8	9	10
M	220	1,184	2,620	5,020	6,232	10,744	12,285	17,296	63,020	66,928
N	284	1,210	2,924	5,564	6,368	10,856	14,595	18,416	76,084	66,992

An amusing historical fact is that the second smallest pair of amicable numbers, 1184 and 1210, was discovered in 1866 by a 16 year old Italian, Nicolò I. Paganini. There are currently 1,227,817,086 known amicable pairs according to the Amicable Pairs List website (accessed 2/16/2023).

Note 3.3.B. Eves on page 77 says that perfect, deficient, and abundant numbers are "sometimes ascribed to the Pythagoreans." One source which makes this connection is W. W. Rouse Ball's *A Short Account of the History of Mathematics*, third edition, MacMillan and Company (1901), which states on its page 30: "As to the work of the Pythagoreans on the factors of numbers we know very little: they classified numbers by comparing them with the sum of their integral subdivisors or factors, calling a number excessive, perfect, or defective according as it was greater than, equal to, or less than the sum of these subdivisors." A more contemporary reference is the MacTutor History of Mathematics Archive page on Perfect Numbers (accessed 2/17/2023) which states: "Perfect numbers were studied by Pythagoras and his followers, more for their mystical properties than for their number theoretic properties." However, in Thomas Heath's more academically credentialed *A History of Greek Mathematics, Volume I. From Thales to Euclid* (Clarendon Press, Oxford, 1921), it is stated on page 74: "There is no trace in the fragments of Philolaus, in Plato or Aristotle, or anywhere before Euclid, of the *perfect* number (*τέλειος*) in the well-known sense of Euclid's definition (VII. Def. 22)..." We'll discuss Euclid's contributions more after we state some formal definitions. These numbers are addressed in Mathematical Reasoning (MATH 3000) in Section 6.9. Perfect Numbers, Mersenne Primes, Arithmetic Functions and in Elementary Number Theory (MATH 3120) in Section 8. Perfect Numbers (a good deal of history is given in both of these sources).

Definition. A positive integer is *perfect* if it is the sum of its proper divisors, *deficient* if it exceeds the sum of its proper divisors, and *abundant* if it is less than the sum of its proper divisors.

Note 3.3.C. We now consider some history of perfect numbers; this history is repeated in the Mathematical Reasoning and Elementary Number Theory notes mentioned above. The MacTutor History of Mathematics Archive's page on "Perfect Numbers" (on which most of this history is base; accessed 2/17/2023) mentions that it is not known as to when perfect numbers were first studied, but suggests that the Egyptians may have been aware of this idea (the page cites C. M. Taisbak's "Perfect numbers: A mathematical pun? An analysis of the last theorem in the ninth book of Euclid's Elements," *Centaurus* **20**(4), 269–275 (1976)). The webpage also mentions that Pythagoras took a mystical interest in perfect numbers. The first recorded mathematical result on perfect numbers appears around 300 BCE in Euclid's *Elements* in Book IX as Proposition 36: "If as many numbers as we please beginning from a unit are set out continuously in double proportion until the sum of all becomes prime, and if the sum multiplied into the last makes some number, then the product is perfect." In modern terminology, this translates into the claim: "If for some $k > 1$ we have $2^k - 1$ prime, then $2^{k-1}(2^k - 1)$ is a perfect number." The condition $2^k - 1$ is prime requires that k itself is prime (see Exercise 6.93 of my Mathematical Reasoning notes on 6.9. Perfect Numbers, Mersenne Primes, Arithmetic Functions). A prime number of the form $2^k - 1$ is a *Mersenne prime*, named after the seventeenth century monk Marin Mersenne (for more details, see Section 10.6. Some Seventeenth-Century Mathematicains of France and Italy). Around 100 CE Nichomachus of Gerasa (circa 60 CE–120 CE) in his *Introductio Arithmetica* (a foundational work in classical algebra) gives a classification of numbers based on the idea of perfect numbers. By adding up what was called the "aliquot parts" of a number (what we call "proper divisors" of the number), he classified numbers (i.e., positive integers) as deficient, *superabundant* (what we call "abundant"; we'll see a different definition of superabundant below), and perfect. This idea of some type of "balance" with perfect numbers has been taken up by some in the religious and mystical community (Nichomachus himself made some strange observations). Nichomachus made several claims about perfect numbers, but provided no proofs. Some of his claims are true, some are false, and some are still open problems. In particular, he claimed that there are infinitely many perfect numbers. This and his other claims are bold, given that there were only four perfect numbers known at the time: 6, 28, 496, and 8128. According to the Wikipedia page "List of Mersenne Primes and Perfect Numbers" (accessed 2/17/2023), there are 51 known Mersenne primes and perfect numbers. The largest known perfect number is $2^{p-1}(2^p - 1)$ where $p = 82{,}589{,}933$, computed in late 2018; it has almost 50 million digits (more details can be found on the Great Internet Mersenne Prime Search (GIMPS) website) and $2^{82{,}589{,}933} - 1$ presently (February 17, 2023) stands as the largest known prime number (it has almost 25 million digits). Unsolved problems that exist to this day include:

1. Are there infinitely many perfect numbers?

2. Are there infinitely many Mersenne primes?

3. Are there any odd perfect numbers?

Note. If adopt the standard notation of $\sigma(n)$ as the sum of all of the (positive) divisors of n (including n itself), that is $\sigma(n) = \sum_{d \mid n} d$, then $n \in \mathbb{N}$ is perfect if and only if $\sigma(n) = 2n$ (since for n perfect, the sum of the proper divisors is n). We can also define $n \in \mathbb{N}$ as *k-tuply perfect* is $\sigma(n) = kn$ for some $k \in \mathbb{N}$. We then have that the 2-tuply perfect numbers are precisely the perfect numbers. Examples of 3-tuply perfect numbers are 120 (since 120 had the divisors 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, and 120, which sum to $360 = 3 \times 120$) and 672. It is unknown if there are infinitely many multiply perfect numbers or if an odd multiply perfect number exists. The Wikipedia page on Multiply Perfect Numbers lists the smallest known k-perfect numbers for $2 \leq k \leq 11$ (accessed 2/17/2023). The smallest known 10-tuply perfect number has 639 digits!

Note. A natural number $n \in \mathbb{N}$ is *superabundant* if $\sigma(n)/h \geq \sigma(k)/k$ for all $2 \leq k \leq n$. These were introduced in Leonidas Alaoglu and Paul Erdős in "On Highly Composite and Similar Numbers," *Transactions of the American Mathematical Society*, **56**(3), 448–469 (1944). A copy can be viewed online on the JSTOR website. It is known that there are infinitely many superabundant numbers (this follows from Alaoglu and Erdős' Theorem 9). This illustrates that some of the ideas from as far back as Euclid's time (and possibly even Pythagoras' time) affect the direction of modern number theory research.

Note. Since this section is on "Pythagorean Arithmetic," we now shift to a topic more confidently affiliated with Pythagoras and the Pythagorean, the *figurate numbers* (or "figured numbers"). These result from arranging a collection of dots (or pebbles or seeds) in regular geometric pattern. Figures 7, 8, and 9 illustrate this idea for triangular numbers, square numbers, and pentagonal numbers.

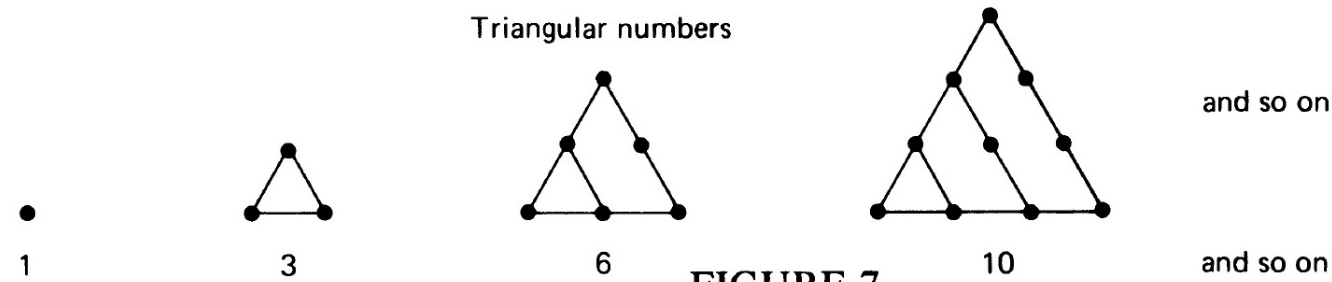

FIGURE 7

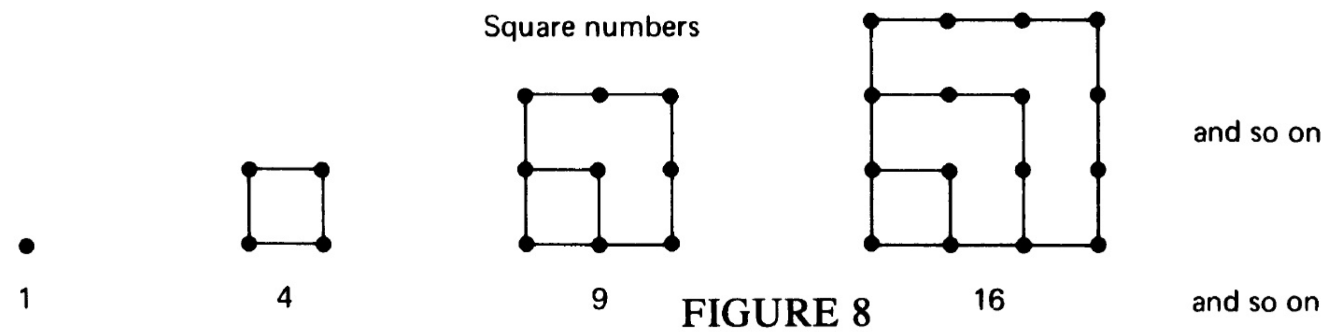

FIGURE 8

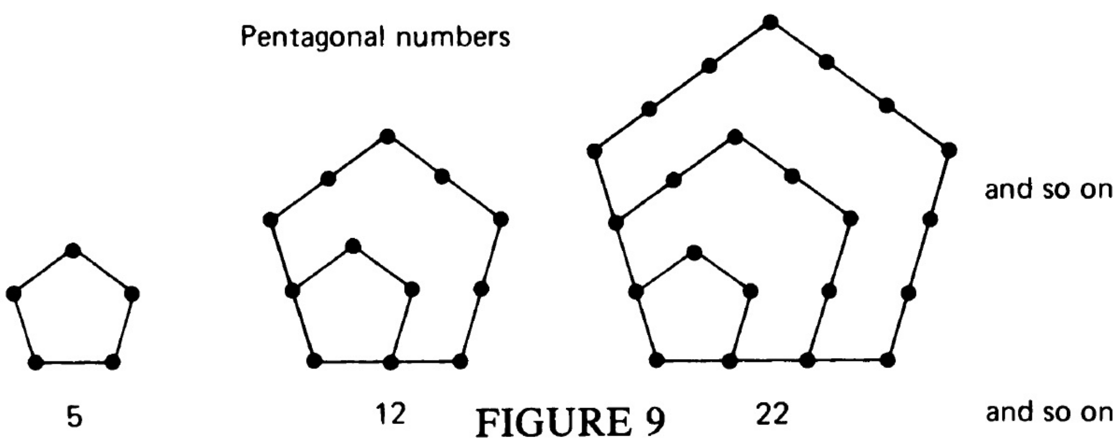

FIGURE 9

Note. Thomas Heath in *A History of Greek Mathematics, Volume I. From Thales to Euclid* states on page 76: "It seems clear the oldest Pythagoreans were acquainted with the formation of triangular and square numbers by means of pebbles or dots." A footnote references Aristotle's *Metaphysics* "1092b." This seems to appear in Book 14 of *Metaphysics*, which is online on the Tufts University's Perseus Digital Library. At [1090a] (this seems to be some type of indexing; it is posted within the online text) Aristotle writes: "The Pythagoreans, on the other hand, observing that many attributes of numbers apply to sensible bodies, assumed that real things are numbers; not that numbers exist separately, but that real things are composed of numbers. But why? Because the attributes of numbers are to be found in a musical scale, in the heavens, and in many other connections." A [1092b] we have: "Eurytus [a disciple of Philolaus] determined which number belongs to which thing—e.g. this number to man, and this to horseby using pebbles to copy the shape of natural objects, like those who arrange numbers in the form of geometrical figures, the triangle and the square." Heath also states on page 76: "It was probably Pythagoras who discovered that the sum of any number of successive terms of the series of natural numbers 1, 2, 3, ... beginning from 1 makes a triangular number." This is illustrated in Figure 7.

Note 3.3.A. As shown in Calculus 1 (MATH 1910; see my online Calculus 1 notes on Section 5.2. Sigma Notation and Limits of Finite Sums and notice Theorem 5.2.B; a proof is given in Appendix A.2. Mathematical Induction in Example A.2.1), the nth triangular number T_n is

$$T_n = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

From Figure 9 we see that the nth pentagonal number P_n is

$$P_n = 1 + 4 + 7 + \cdots + (3n-2) = \sum_{i=1}^{n}(3i-2) = 3\left(\sum_{i=1}^{n} i\right) - 2n$$

$$= 3\frac{n(n+1)}{2} - 2n = \frac{3n(n+1) - 4n}{2} = \frac{3n^2 - n}{2} = \frac{n(3n-1)}{2}.$$

This gives us the equipment to algebraically establish relationships between various figurate numbers.

Note. Next, we present three theorems and then give a geometric argument (using a figure) and an algebraic argument for the validity of each.

Theorem I. Any square number is the sum of two successive triangular numbers.

Proofs. A geometric proof is given in Figure 10 below. For an algebraic proof, let S_n denote the nth square number, so that $S_n = n^2$. To show the theorem, we have

$$S_n = n^2 = \frac{(n^2+n) + (n^2-n)}{2} = \frac{n(n+1)}{2} + \frac{(n-1)n}{2} = T_n + T_{n-1},$$

by Note 3.3.A. That is, square n^2 is the sum of the consecutive triangular numbers T_{n-1} and T_n, as claimed. ∎

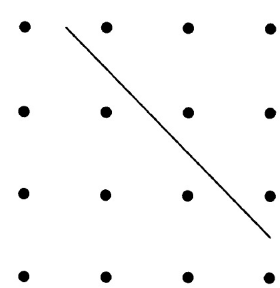

FIGURE 10

Theorem II. The nth pentagonal number is equal to n plus three times the $(n-1)$th triangular number.

Proofs. A geometric proof is given in Figure 11 below. For an algebraic proof, the

nth pentagon number is

$$P_n = \frac{n(3n-1)}{2} = \frac{3n^2-n}{2} = \frac{3n^2-3n+2n}{2} = n + \frac{3n(n-1)}{2} = n + 3T_{n-1}.$$

That is, P_n equals n plus three times the $(n01)$th triangular number T_{n-1}. ∎

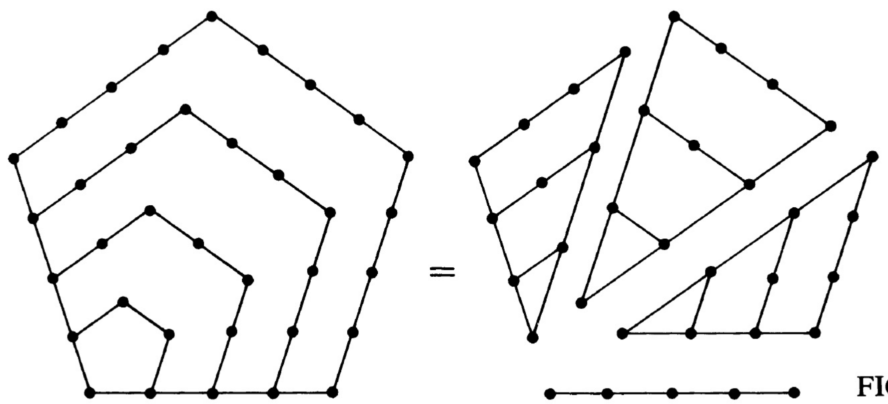

FIGURE 11

Theorem III. The sum of any number of consecutive odd integers, starting with 1, is a perfect square.

Proofs. A geometric proof is given in Figure 12 below. For an algebraic proof, we have

$$1+3+5+\cdots+(2n-1) = \sum_{i=1}^{n}(2i-1) = 2\left(\sum_{i=1}^{n} i\right) - n = 2\frac{n(n+1)}{2} - n = n^2,$$

as claimed. ∎

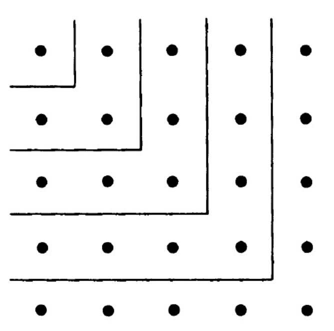

FIGURE 12