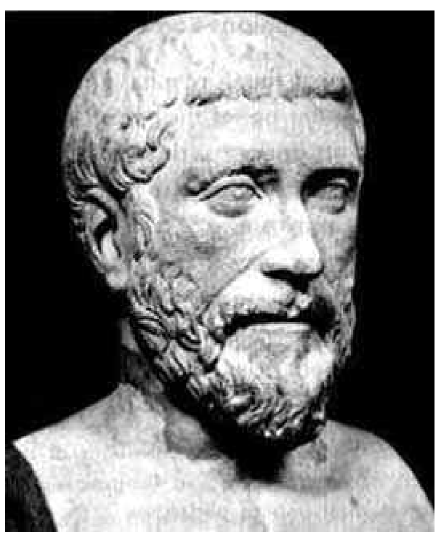
3.4. Pythagorean Theorem and Pythagorean Triples

Note. In this brief section, we discuss Pythagoras' Theorem: "The square of the hypotenuse of a right triangle is equal to the sum of the squares of the two legs." We observe that this result was known to the Babylonians and mention the type of proof probably given by Pythagoras. Pythagorean triples are defined and classified in modern terms.

Note. We saw in Section 2.4. Babylonia: Geometry and Section 2.6. Babylonia: Plimpton 322 that the Pythagorean Theorem was known more than a thousand years before Pythagoras' time (according to the MacTutor History of Mathematics Archive biography of Pythagoras of Samos, Pythagoras lived from about 570 BCE to about 490 BCE).



Pythagoras of Samos (circa 570 BCE to circa 490 BCE)

Image from the MacTutor History of Mathematics Archive biography of Pythagoras of Samos

Note. According to Eves (see page 81) "the first first general proof of the theorem may be been given by Pythagoras." He thinks that Pythagoras knew a dissection type of proof, like in Figure 13.

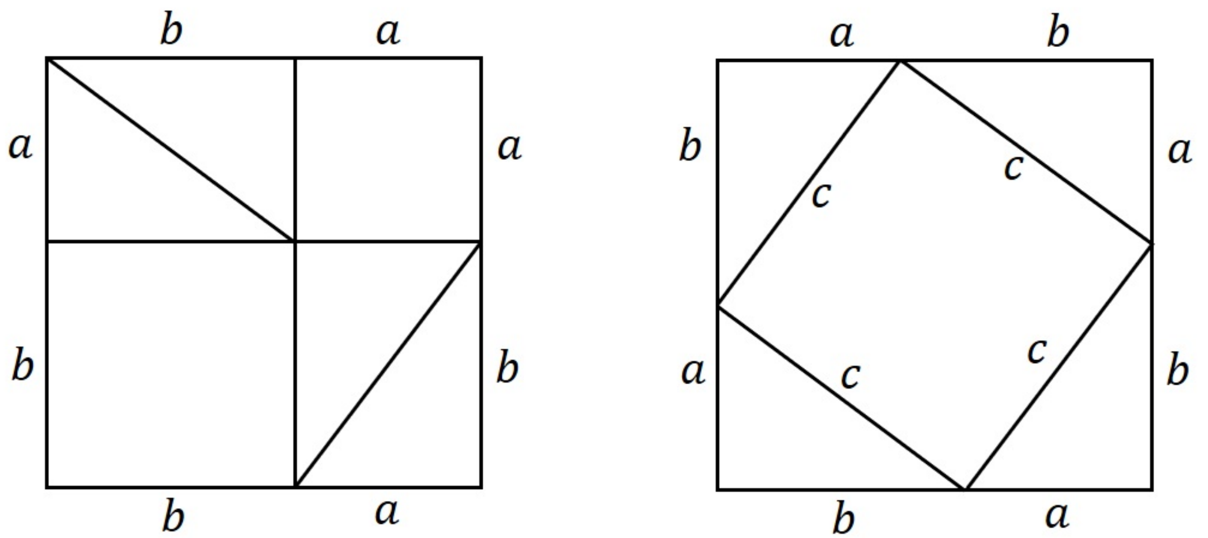


Figure 13 (modified slightly)

We see in Figure 13 left that the area of the square is $a^2 + b^2 + 2ab$ and in Figure 13 right that the area of the square is $4(ab/2) + c^2 = 2ab + c^2$. Therefore, $a^2 + b^2 = b^2$. Now underlying these observations are facts concerning the sums of angles in a triangle and some knowledge of parallels. Eves observes that (page 81) "the early Pythagoreans are also credited with the development of that theory." A more in-depth history of the Pythagorean Theorem (including a discussion of Euclid's proof in the *Elements*) can be found in my online notes for Introduction to Modern Geometry-History (MATH 4157/5157) on Section 1.7. The Pythagorean Theorem where Chinese, Indian, and Arabic proofs are discussed. A webpage with animations illustrating these three proofs, and many others, is available on the Many Proofs of Pythagorean Theorem webpage (accessed 2/11/2022).

Note. Thomas Heath in *A History of Greek Mathematics, Volume I: From Thales to Euclid* states (see page 144): "Though this [the Pythagorean Theorem] is the proposition universally associated by tradition with the name of Pythagoras, no really trustworthy evidence exists that it was actually discovered by him. The comparatively late writers [including Proclus and Plutarch] who attribute it to him add the story that he sacrificed an ox to celebrate his discovery." He also comments that (see page 148): "...it would appear most probable that Pythagoras would prove the proposition by means of his (imperfect) theory of proportions." We'll discuss the the proof of the Pythagorean Theorem as presented by Euclid in his elements in "Chapter 5. Euclid and His Elements."

Note/Definition. Related to the Pythagorean theorem is the idea of finding (positive) integers a, b, c such that $a^2 + b^2 = c^2$. Such a trio of integers a, b, c is called a *Pythagorean triple*. As seen in Section 2.6. Babylonia: Plimpton 322, the Babylonians knew how to calculate such triples. The Pythagoreans have been credited with the formula

$$m^2 + \left(\frac{m^2-1}{2}\right)^2 = \left(\frac{m^2+1}{2}\right)^2,$$

where m is any odd integer, which produces a Pythagorean triple. The similar formula (which results by multiplying the previous formula by 4 on both sides) $(2m)^2 + (m^2-1)^2 = (m^2+1)^2$, where m may be even or odd, is attributed to Plate (circa 380 BCE). But neither of these formulae give all Pythagorean triples.

Note. For more details on Pythagorean triples, see my online notes for Elementary Number Theory (MATH 3120) on Section 16. Pythagorean Triangles. In these notes, all Pythagorean triples are given. It is shown that a Pythagorean triple a, b, c of positive integers where a, b, c have no common factor and a is even are given by $a = 2mn$, $b = m^2 - n^2$, and $c = m^2 + n^2$ where m and n are any relatively prime integers, not both odd, and $m > n$ (see Theorem 16.1).

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