3.5. Discovery of Irrational Magnitudes

Note. In this section we give the history of the "discovery" of the irrational numbers (or "incommensurable lengths," as it would be stated at the time), as it is understood. The Pythagoreans are center to the study, though the historical record is incomplete.

Note. The positive integers (or "natural numbers," $\mathbb{N}$) have their roots in the simple act of counting. Fractions follow naturally from this when measuring certain things (or weighing things). This leads to the positive rational numbers. **Ratio**nal because they are ratios; the positive rational numbers are of the form p/q where $p, q \in \mathbb{N}$. The integer zero enters as a number concept sometime around 500 CE (we will explore this in detail later). The wide acceptance of negative numbers comes about with the growth of the theory of equations comes about in the 1500s, which we'll discuss in Section 8.8. Cubic and Quartic Equations. Combining these ideas, we get the mathematical structures of the *integers* (both positive, negative, and zero) $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ and the *rational numbers* $\mathbb{Q} = \{p/q \mid p, q \in \mathbb{Z}, q \neq 0\}$. We use the symbol $\mathbb{Z}$ for the integers from the German word for "count," *zalen*. We use the symbol $\mathbb{Q}$ for the rationals because they are *quotients* of integers.

Note. Eves describes (on page 82) a geometric interpretation of rational numbers. Starting with a given line segment which we take to define a unit of length, we mark out repeated copies of this length on a line to produce line segments of any length n where $n \in \mathbb{N}$. Using a compass and straight edge, we can then subdivide a unit length segment into n pieces of equal length, yielding a line segment of length $1/n$ for each $n \in \mathbb{N}$. By taking m repeated copies of such a segment, we can construct a line segment of length m/n for any $m, n \in \mathbb{N}$. That is, we can construct a segment of and length $q > 0$ where $q \in \mathbb{Q}$. These ideas are addressed in Introduction to Modern Algebra 2 (MATH 4137/5137). See my online notes for this class on Section VI.32. Geometric Constructions (notice Theorem 32.1). I also have a PowerPoint presentation (with a transcript) and a 24-minute YouTube video on this material ('Compass and Straight Edge Constructions").

Note 3.5.A. The geometric construction of rational numbers was thought to be a way to describe all points on the number line (though the number line is a much later idea, introduced by Descartes in the early 1600s). The Pythagoreans are credited with the discovery that there are, in fact, numbers that can be constructed that are *not* rational. That is, the Pythagoreans proved the existence of *irrational* numbers. They did so by showing that the length of the diagonal of a square with sides of length one, is not rational. That is, $\sqrt{2}$ is irrational. A quick comment about rational and irrational numbers is appropriate here. We have by definition, that the integers are the natural numbers, their negatives, and 0: $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$. The rational numbers are then $\mathbb{Q} = \{p/q \mid p, q \in \mathbb{Z}, q \neq 0\}$. Denoting the real numbers as $\mathbb{R}$, the irrational numbers are $\mathbb{R} \setminus \mathbb{Q}$ where "$\setminus$" represents set subtraction). Of course without a clear definition of the real numbers, the irrational numbers remain nebulous! You may have heard that a rational number is one for which the decimal expansion either ends or repeats, and an irrational number is one whose decimal expansion neither terminates nor repeats. Though true, the definition in terms of ratios is more to the point; the decimal expansion approach likely stems from the ubiquitous use of electronic calculators starting in the 1970s. The topic of repeating decimals is covered in Elementary to Number Theory (MATH 3120) in Section 15. Decimals (see, in particular, Theorem 15.3).

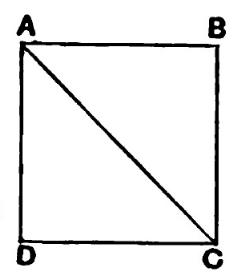
Note 3.5.B. We know by the Pythagorean Theorem that the diagonal of a unit square has length $s = \sqrt{2}$ (so that $\sqrt{2}$ is "constructible" with a straight edge and compass). ASSUME $\sqrt{2}$ is rational, say $\sqrt{2} = a/b$ for some $a, b \in \mathbb{N}$. Choose such a and b so that they are relatively prime; that is, they have no common factors (and the expression a/b is in "lowest terms"). This can be done by the Fundamental Theorem of Arithmetic, also covered in Elementary to Number Theory (MATH 3120); see Section 2. Unique Factorization and notice Theorem 2.2. Then $a = b\sqrt{2}$ or, squaring both sides (we take a and b positive, and of course $\sqrt{2}$ is positive because that is what the square root notation means, so we have no concerns over extraneous roots as we proceed) then $a^2 = 2b^2$. We now see that a^2 is even and, since 2 is prime, we must have that a itself is even, say $a = 2c$. This gives $a^2 = 4c^2 = 2b^2$ or $2c^2 = b^2$. But now we have that b^2 is even and so b itself is even. But this is a CONTRADICTION to the fact that we chose a and b to be relatively prime. So the assumption that $\sqrt{2}$ is rational must be false, and hence $\sqrt{2}$ is irrational. A similar argument can by used to show that $\sqrt{p}$ is irrational for any prime p (see Problem 3.7(c)).

Note 3.5.C. A geometric interpretation of these ideas can be viewed by considering two given line segments, ℓ_1 and ℓ_2. We might search for a third line segment ℓ_3 such that both ℓ_1 and ℓ_2 have lengths that are a natural number multiple of the length of ℓ_3. Such line segments ℓ_1 and ℓ_2 were called by the Greeks *commensurable*. If there is no such third line segment ℓ_3, the line segments ℓ_1 and ℓ_2 are *incommensurable*. We now have that the diagonal of unit square is incommensurable with the unit length side of the square, as just argued. Sir Thomas Heath in his translation into English of Euclid's work, *The Thirteen Books of Euclid's Elements* (Cambridge University Press, 1908), Volume III (containing Books X–XIII), states (see page 2):

> "The actual method by which the Pythagoreans proved the incommensurability of $\sqrt{2}$ with unity was no doubt that referred to be Aristotle (*Anal. prior.* I.23, 41 a 25–7), a *reductio ad absurdum* by which it is proved that, if the diagonal is commensurable with the side, it will follow that the same number is both odd and even."

Heath then presents the exact same argument as given above. In this, a colon ':' represents division (i.e., a quotient or ratio):

Suppose AC, the diagonal of a square, to be commensurable with AB, its side. Let $\alpha : \beta$ be their ratio expressed in the smallest numbers.



Then $\alpha > \beta$ and therefore necessarily > 1.

Now $$AC^2 : AB^2 = \alpha^2 : \beta^2,$$

and, since $$AC^2 = 2AB^2, \qquad \text{[Eucl. I. 47]}$$

$$\alpha^2 = 2\beta^2.$$

Therefore α^2 is even, and therefore α is even.

Since $\alpha : \beta$ is in its lowest terms, it follows that β must be *odd*.

Put $$\alpha = 2\gamma;$$

therefore $$4\gamma^2 = 2\beta^2,$$

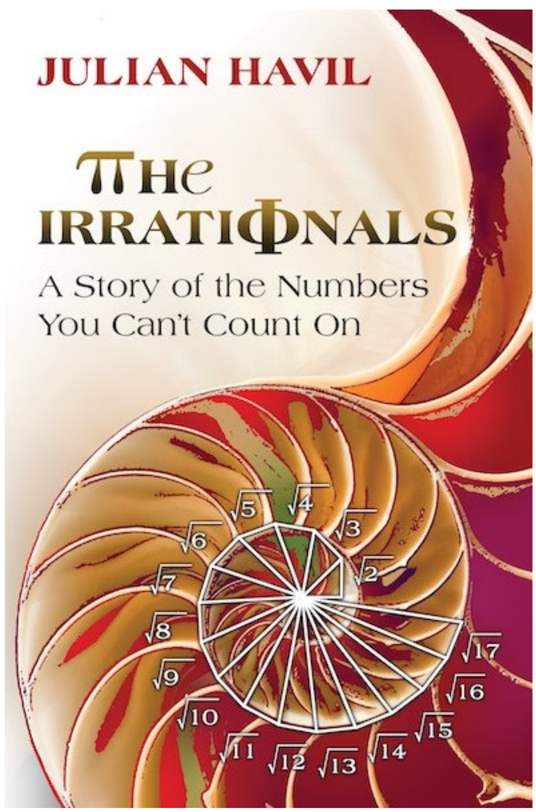
or $$\beta^2 = 2\gamma^2,$$

so that β^2, and therefore β, must be *even*.

But β was also odd:

which is impossible.

Note. According to Thomas Heath's *A History of Greek Mathematics, Volume I: From Thales to Euclid* (Oxford University Press, 1921; see Page 155), Plato states in *Theaetetus* (a dialogue between Socrates and mathematician Theaetetus, considered a founding work on epistemology) that Theodorus of Cyrene (around 425 BCE) showed that $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, $\sqrt{8}$, $\sqrt{10}$, $\sqrt{11}$, $\sqrt{12}$, $\sqrt{13}$, $\sqrt{14}$ $\sqrt{15}$, and $\sqrt{17}$ are irrational. This is explored in Problem 3.15(b), where it is stated that Theodorus *constructed* $\sqrt{n}$ for $2 \le n \le 17$ and it is suggested that he may have used a spiral shape to recursively construct these. The spiral appears on the cover of Julian Havil's book *The Irrationals: A Story of the Numbers You Can't Count On* (Princeton University Press, 2012):



Havil's book gives a thorough popular-level history of the irrational and transcendental numbers (though it contains LOTS of equations, and includes the use of series). The first chapter covers the topic of this section of notes. Appendix A deals with "the spiral of Theodorus" and illustrates how it can be represented in the complex plane $\mathbb{C}$ (going beyond $n = 17$). It is also shown that the function $T : \mathbb{R} \to \mathbb{C}$ defined as $T(\alpha) = \sqrt{\alpha + 1}e^{i\theta}$ produces all of these points and has as its graph a spiral in $\mathbb{C}$ (the Theodorus spiral asymptotically approaches an Archimedean spiral). The parameter α is related to the argument of the complex number $z = T(\alpha)$.

Note 3.5.D. In *The Irrationals: A Story of the Numbers You Can't Count On*, Havil relays the story that the member of the Pythagorean school Hippasus is the one who discovered irrational numbers and that he was punished for it (see Havil's paged 20 and 21):

> Not only is [Hippasus of Metapontum] accused of destroying the concept of commensurability, he is meant to have spoken of the horror outside the secretive Pythagorean community—and he is meant to have done the same with his discovery that a dodecahedron can be inscribed within a sphere. With the authority of Iamblichus of Chalcis [in his *The Life of Pythagoras*]: ...they cast him out of the community; they built a shrine for him as if he were dead, he who had once been their friend. Others add that even the gods became angry with him who had divulged Pythagoras' doctrine; that he who showed how the dodecahedron can be inscribed within a sphere died at sea like an evil man. Others still say that the same misfortune happened on him who spoke to others of irrational numbers and incommensurability."

This story is briefly mentioned by Eves on page 84). Hippasus' connection to pentagons, the dodecahedron, and irrational numbers is also mentioned in my online notes for Introduction to Modern Geometry (MATH 4157/5157) on Section 1.4. The Regular Pentagon. However, the fate of Hippasus is not well-documented in the historical record. Concerning this, Charles Kahn in *Pythagoras and the Pythagoreans: A Brief History* (Hackett Publishing, 2001) states (see his page 35): "The many stories about him [Hippasus] and his punishment for revealing Pythagorean secrets, or for claiming them as his own, sound more like legend than history."

Hippasus engraving by Girolamo Olgiati (1580) from the
Wikipedia page on Hippasus

Note. As discussed in Section 3.2. Pythagoras and the Pythagoreans, the Pythagoreans assigned mystical powers to numbers. They assumed that the everything depends on the whole numbers (as Eves puts it on page 84). They had developed an idea of proportions that depended on commensurable magnitudes. The discovery of incommensurable magnitudes (and hence irrational numbers) would have been a significant blow to the philosophical and religious views of the Pythagoreans. A new theory of proportions was needed.

Note 3.5.E. This new theory is due to Eudoxus (408 BCE–355 BCE) and is largely the content of Euclid's Book V. Thomas Heath in *The Thirteen Books of Euclid's Elements Volume II*, in his Introductory Note to Book V he lays out the history as follows (see his pages 112 and 113):

> "The anonymous author of a scholium [a scholium is an annotated written work with marginal comments or explanations made by a commentator who has studied, but did not author, the work] to Book V . . . , who is perhaps Proclus, tells us that 'some say' this Book containing the general theory of proportion. . . 'is the discovery of Eudoxus, the teacher of Plato.' Not that there had been no theory of proportion developed before his time; on the contrary it is certain that the Pythagoreans had worked out such a theory with regard to *numbers*, by which must be understood commensurable and even whole numbers. . . The discovery of [the theory of proportion independent of commensurability] belongs then most probably to Eudoxus."

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