3.6. Algebraic Identities

Note. In this section, we establish three algebraic identities in terms of areas of rectangles. The results presented appear in Euclid's *Elements*, Book II as Propositions 4, 5, and 6.

Note. The early Greeks were "completely lacking any adequate algebraic notation [and so] devised ingenious geometrical processes for carrying out algebraic operations." (See page of Eves 85.) These processes are part of *geometrical algebra* (the topic of this section and the next in these notes). 'Numbers' conceptually represented quantities (or magnitudes), so negative numbers were nonexistent (though subtraction was allowed, but it was always represented in terms of sums of areas). In fact, zero was not yet a number!

Note. Thomas Heath in *A History of Greek Mathematics, Volume I: From Thales to Euclid* states (see page 150):

> "It is certain that the theory of application of areas [to geometrical algebra] originated with the Pythagoreans, if not with Pythagoras himself. We have this on the authority of Eudemus, quoted in the following passage of Proclus: 'These things, says Eudemus, are ancient, being discoveries of the Must of the Pythagoreans, I mean the *application of areas*. . . , their *exceeding*. . . and their *falling short*. . . . It was from the Pythagoreans that later geometers (i.e. Apollonius of Perga) took the names, which they then transferred to the so-called *conic* lines (curves), calling one of these a *parabola* (application), another a *hyperbola* (exceeding), and the third an *ellipse* (falling short). . . "

We'll explore conic section further in Section 6.4. Apollonius. This is also addressed in Introduction to Modern Geometry (MATH 4157/5157) in Chapter 3. Conic Sections.

Note. Euclid's *Elements*, Books II contain some results on geometric algebra. Books I and VI also include such results related to the quadratic equation; these will be presented in the next section.

Note. The wording of results in Euclid's *Elements* are awkward by contemporary standards. Proposition 4 of Book II states: "If a straight line is divided into any two parts, the square on the whole line is equal to the sum of the squares on the two parts together with twice the rectangle contained by the two parts." With the lengths of the "two parts" as a and b, the "sum of the squares" is $a^2 + b^2$ (literally, the areas of two *squares* with sides of lengths a and of length b). The "twice the rectangle contained by the two parts" quantity is then $2ab$. Algebraically, the proposition is $(a + b)^2 = a^2 + 2ab + b^2$. Geometrically, it is as given in Figure 16:.

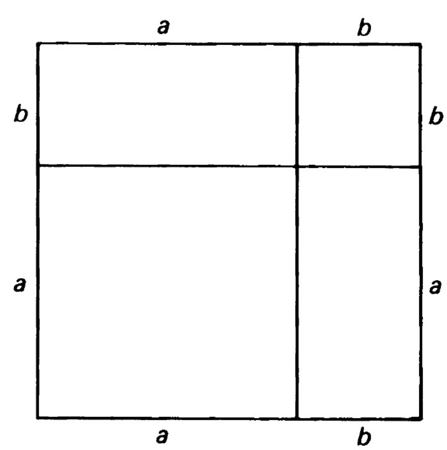


Figure 16

Note. Proposition 5 of Book II of Euclid is: "If a straight line is divided equally and also unequally, the rectangle contained by the unequal parts, together with the square on the line between the point of section, is equal to the square on half the line." Now, "divided equally" means that the segment is bisected (that is, cut into two equal length pieces), whereas "and also unequally" means that another point on (and interior to) the line segment is chosen. In Figure 17 below (based on the argument given by Euclid), the line segment if segment $\overline{PB}$, with bisecting point P and "also unequally" point Q.

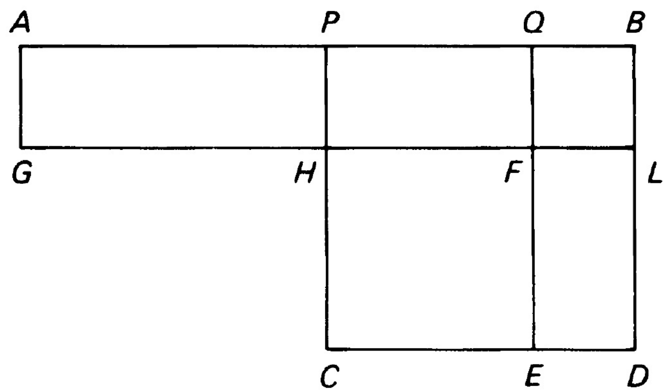
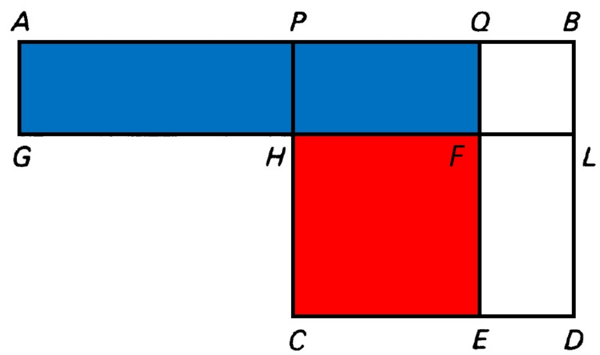
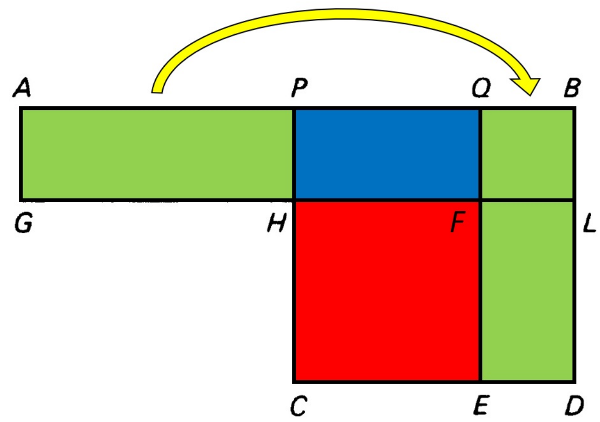


Figure 17

As in Eves, we represent lengths of line segments using pairs of endpoints of the segment, so that we have AP is the length of line segment $\overline{AP}$ and we have (because of the bisection) that $AP = PB$. The figure is to be interpreted as having $PQ = HF = CE = HC = FE$, so that points H, C, E, F form a square. Eves represents the area of this square as $HCEF$ so that $(PQ)^2 = HCEF$. We also have in Figure 17 that $QB = FL = QF = BL$, $AG = PH = QF = BL$, $FL = ED$, $FE = LD$, and so $PB = PQ + QB = FE + BL = BL + LD = BD$ (notice that this is possible if we take all interior angles to be right angles; this follows from the Parallel Postulate). Next, $AP = PB$ since point P results in a bisection of $\overline{AB}$. Therefore, we have that $AGHP = (AP)(AG) = (PB)(BL) = (BD)(QB) = QEDB$ (as is reflected in green in the figure below, right). Hence $(AQ)(QB) + (PQ)^2 = (PB)^2$.





Alternatively (as Eves describes) we can decompose the areas into the following rectangles and squares:

$$\begin{aligned} (AQ)(QB) + (PQ)^2 &= AGFQ + HCEF = AGHP + PHFQ + HCEF \\ &= PHLB + PHFQ + HCEF \\ &= PHLB + FEDL + HCEF = (PB)^2. \end{aligned}$$

If we set $AQ = 2a$ and $QB = 2b$, then $AB = AQ + AB = 2a + 2b$ and so $AB = PB = a + b$. Also $PQ = PB - QB = (a + b) - (2b) = a - b$. Then $(AQ)(QB)+(PQ)^2 = (PB)^2$ becomes $(2a)(2b)+(a-b)^2 = (a+b)^2$ or $4ab+(a-b)^2 = (a + b)^2$. Alternatively, if we set $AB = 2a$ (so that $AP = PB = a$) and $PQ = b$, then $QB = PB - PQ = a - b$ and $AQ = AB - QB = (2a) - (a - b) = a + b$. Then $(AQ)(QB) + (PQ)^2 = (PB)^2$ becomes $(a+b)(a-b) + (b)^2 = (a)^2$ or $(a+b)(a-b) = a^2 - b^2$. So we get the two familiar algebraic identities

$$4ab + (a - b)^2 = (a + b)^2 \text{ and } (a + b)(a - b) = a^2 - b^2.$$

Note. Proposition 6 of Book II of Euclid is: "If a straight line is bisected and produced to any point, the rectangle contained by the whole line thus produced and the part of it produced, together with the square on half the line bisected, is equal to the square on the straight line made up of the half and the part produced." Let the "straight line" be line segment $\overline{AB}$, let it be bisected at point P, and extend $\overline{AB}$ be "produced to" point Q. Then "the rectangle contained by the whole line thus produced [that is, line segment $\overline{AQ}$] and the part of it produced [that is, line segment $\overline{BQ}$]" is the rectangle with area $(AQ)(BQ)$, and the "square on half the line bisected" is a square with sides of length PB. This gives an area of $(AQ)(BQ)+(PB)^2$. This is claimed to be equal to "the square on the straight line made up of the half and the part produced" is the line segment $\overline{PQ}$. See Figure 18 below. Proposition is then making the claim that $(AQ)(BQ) + (PB)^2 = (PQ)^2$. This can be justified with the same "dissection" of the areas given in Figure 17 for Proposition, but with B and Q interchanged. Notice that with $AQ = 2a$ and $BQ = 2b$, we again have $4ab + (a-b)^2 = (a+b)^2$.

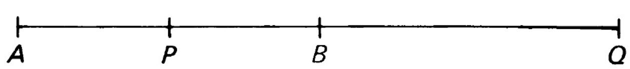


FIGURE 18

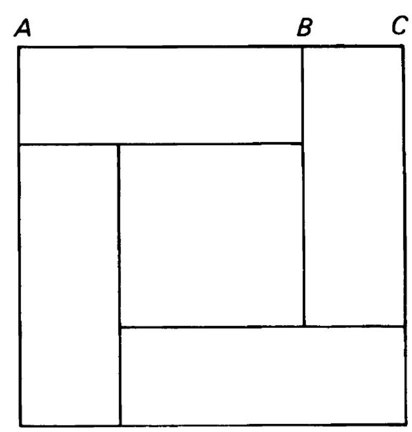


FIGURE 19

Note. A easier use of geometric algebra to establish the identity $4ab + (a - b)^2 = (a + b)^2$ is given by the square in Figure 19 above.

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