3.7. Geometric Solution of Quadratic Equations

Note. In this section, we consider compass and straight edge constructions of solutions to certain quadratic equations. We use similar triangles, circles, and parallelograms to construct solutions to quadratics of the form $x^2 = ab$, $x^2 - ax + b^2 = 0$, and $x^2 - ax - b^2 = 0$, and $x^2 + ax + b^2 = 0$ where a and b are constructible lengths. We regularly appeal to Euclid's *Elements* Book I for needed constructions.

Note. If line segments of lengths a, b, and c are constructible, then by the "method of proportions" a line segment of length x can be constructed satisfying $a : b = c : x$ (using colons to represent ratios). That is, the equation $a/b = c/x$ or $ax = bc$ can be "solved." The construction is given in Figure 20 (left) and it is based on the construction of a line parallel to a given line through a point not on the line (such a construction is given in Euclid Book I as Proposition 31). The fact that the resulting x satisfies the desired equation follows by considering similar triangles. A modern algebra compass and straight edge construction of a solution to $ax = bc$ is given in Introduction to Modern Algebra 2 (MATH 4127/5127) in Section VI.32. Geometric Constructions (see Theorem 32.1); a video containing this material is posted on my YouTube channel as well: Compass and Straight Edge Constructions (see Theorem 32.1 in the video; accessed 3/4/2023). A line segment of length x can also be constructed satisfying $a : x = x : b$. That is, the equation $a/x = x/b$ or $x^2 = ab$ can be "solved." The construction requires a circle with diameter $a + b$, as given in Figure 20 (right). With line segments of length a and b, it is easy enough to construct a segment of length $a + b$. This segment can be bisected (Euclid's Proposition 10 in Book I gives the construction), and then the center and radius of the circle is known so that the circle can be constructed. A perpendicular to the radius at the point where the two line segments meet can be constructed by Euclid Book I Proposition 11. This then determines the segment of length x. We now give a "modern" argument that Figure 20 gives the correct value of x. If center of the circle is at the origin of the Cartesian (X, Y)-plane with the given radius along the X-axis, then the equation of the circle is $X^2 + Y^2 = ((a+b)/2)^2$. We denote the distance of the desired line segment from the origin as X. Assuming, as in Figure 20, that it lies to the right of the center and that $a > b$, then we have $X = (a+b)/2 - b = (a-b)/2$. With the length of the segment denoted $Y = x$ in Figure 20, we have the relationship $X^2+Y^2 = ((a+b)/2)^2$ or $((a-b)/2)^2 + x^2 = ((a+b)/2)^2$ or $(a^2 - 2ab - b^2)/4 + x^2 = (a^2 + 2ab + b^2)/4$ or $x^2 = 2ab/4 + 2ab/4 = ab$, as desired. For the geometric purist, this can also be shown using similar triangles. This argument is in the above-mentioned YouTube video (see Theorem 32.6′).

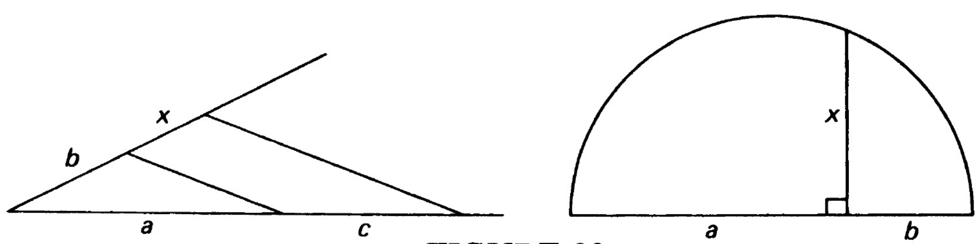


FIGURE 20

Note/Definition. Before we address quadratics as presented by Euclid, we need to translate his wording into pictures to better understand his claims. We are interested in "applying parallelograms to segments." Consider Figure 21.

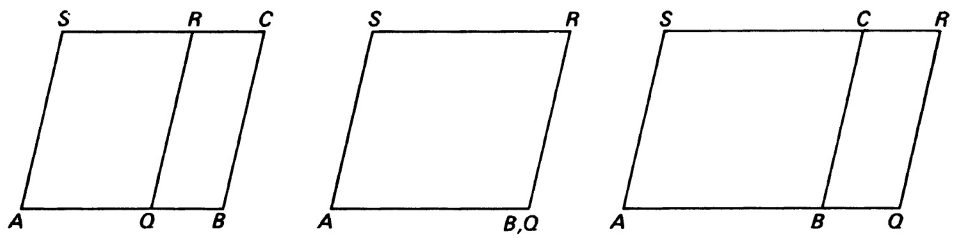


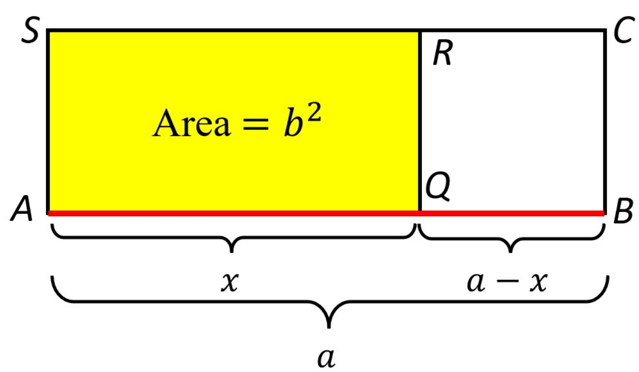
FIGURE 21

We apply parallelogram $AQRS$ to segment AB in three cases (as given in Figure 21 from left to right): (1) when Q is between A and B, (2) when Q and B coincide, and (3) when B is between A and Q. We start with segment $\overline{AB}$ and extend the segment beyond point B (if necessary) and take point Q on the resulting ray. We are then interested in parallelogram $AQRS$ and its relation to the parallelogram $QBCR$. When Q is between A and B, parallelogram $AQRS$ is *applied to segment AB, falling short by Parallelogram $QBCR$.* When A coincides with B, parallelogram $AQRS$ is *applied to segment AB.* When B lies between A and Q, parallelogram $AQRS$ is *applied to segment AB, exceeding by parallelogram $QBCR$.* We will apply parallelogram $AQRS$ of given area to line segment $\overline{AB}$ of known length. In this way, we introduce a quadratic equation concerning the equality of areas of the given parallelogram and the new parallelogram with $\overline{AB}$ as once side. For simplicity, the examples given below involve special cases where the parallelograms are rectangles.

Note 3.7.A. Euclid's Book I Proposition 44 solves the construction: To apply to a given line segment $\overline{AB}$ a parallelogram of given area and given base. (Heath's translation of the Euclid's *Elements* states Proposition 44 as: "To a given straight line to apply, in a given rectilinear angle, a parallelogram equal to a given triangle.") Here we consider the special case in which the base angles are right angles, so that both parallelograms are rectangles. Let the dimensions of the parallelogram (rectangle in our case) $AQRS$ of given area be b and c, so that the given area is bc. With segment $\overline{AB}$ of length, say a, Proposition 44 tells us that applying rectangle $AQRS$ to $\overline{AB}$ yields a rectangle with base $\overline{AB}$ of the same area. Let x denote the altitude of this rectangle. Since the areas are the same, we have $ax = bc$ or $x = bc/a$. So for given line segments of lengths a, b, and c, Proposition 44 gives the existence of a line segment of length bc/a.

Note. If we think in terms of constructible numbers, then constructing $x = bc/a$ is addressed in Introduction to Modern Algebra 2 (MATH 4137/5137) in Section VI.32. Geometric Constructions (notice Theorem 32.1); we also mentioned this in Section 3.5. Discovery of Irrational Magnitudes.

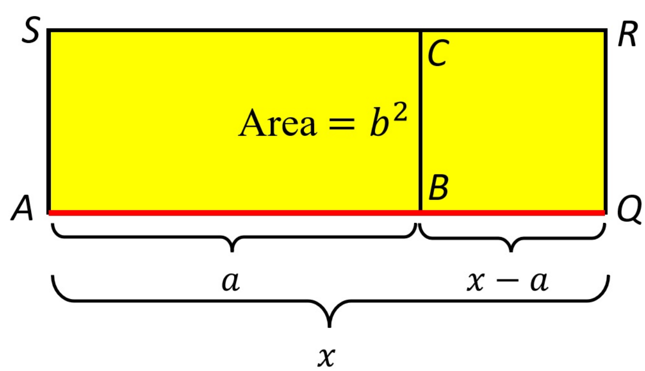
Note 3.7.B. Proposition 28 of Euclid's Book VI solves the construction: To apply to a given line segment $\overline{AB}$ a parallelogram $AQRS$ equal in area to a given rectilinear figure F, and falling short by a parallelogram $QBCR$ similar to a given parallelogram, the area of F not exceeding that of the parallelogram described on half of AB and similar to the defect $QBCR$. This is pretty much Heath's translation as given in his *Elements* (Volume 2), except that the letters are not given in the translation. Consider the special case where the given rectilinear figure F is a square, and the base angles are right angles. To clarify what is being claimed, we turn to Heath's translation of Euclid's *Elements*, Volume 2 which includes commentary on this proposition in the special case. He states (see his page 265): "This is the problem of *applying to a given straight line a rectangle equal to a given area and falling short by a square....*" So we consider Figure 21 (left) where parallelogram $QBCR$ is a square and rectangle $AQRS$ has the area of rectilinear figure F (see the figure below). Denote the length of $\overline{AB}$ as a, and the length of a side of F as b (so that the area of F is b^2). Denote the length of segment $\overline{AQ}$ as x. Then the length of $\overline{QB}$ is the length of $\overline{AB}$ minus the length of $\overline{AQ}$: $a - x$. Since $QBCR$ is a square, then the length of $\overline{BC}$ (and also of $\overline{AS}$) is $a-x$; by giving us a square for the part by which we "fall short," this determines the height of rectangle $AQRS$. Therefore, the area of rectangle $AQRS$ is $x(a - x)$.



Propositions 28 of Book VI tells us that this area equals the area of rectilinear figure F. That is, $x(a - x) = b^2$. This allows us to geometrically solve the quadratic:

$$x(a-x) = b^2 \text{ or } x^2 - ax + b^2 = 0. \tag{1}$$

Note 3.7.C. Proposition 29 of Euclid's Book VI solves the construction: To apply to a given line segment $\overline{AB}$ a parallelogram $AQRS$ equal in area to a given rectilinear figure F, and exceeding by a parallelogram $BQRC$ similar to a given parallelogram. Consider the special case in which the base angles are right angles and the excess is a square. To clarify, we again turn to Heath's translation of Euclid's *Elements*, Volume 2 in which he states (see his page 267): "This is the problem of *applying to a given straight line a rectangle equal to a given area and extending it by a square*." So we consider Figure 21 (right) where parallelogram $BQRC$ is a square and rectangle $AQRS$ has the area of rectilinear figure F (see the figure below). Denote the length of $\overline{AB}$ as a, and the length of a side of F as b (so that the area of F is b^2). Denote the length of segment $\overline{AQ}$ as x. Then the length of $\overline{BQ}$ is the length of $\overline{AQ}$ minus the length of $\overline{BQ}$: $x-a$. Since $BQRC$ is a square, then the length of $\overline{BC}$, $x-a$, is the height of rectangle $AQRS$. Therefore, the area of rectangle $AQRS$ is $x(x-a)$.



Propositions 29 of Book VI tells us that this area equals the area of rectilinear figure F. That is, $x(x-a) = b^2$. This allows us to geometrically solve the quadratic:

$$x(x-a) = b^2 \text{ or } x^2 - ax - b^2 = 0. \tag{2}$$

Note 3.7.D. Eves mentions (as does Heath in his comments; Eves' main source in this section is probably Heath's translation of the *Elements*, Volume 2) that the special case of Proposition 28 of Book VI (which we denote as "Proposition VI.28") as stated in Note 3.7.B, can be addressed with a more elementary approach than that given in Book VI. Euclid's Proposition II.5 states: "If a straight line segment $\overline{AB}$ is cut into equal and unequal segments, then the rectangle contained by the unequal segments of the whole together with the square on the straight line between the points of section is equal to the square on the half." Eves clarifies this as (page 89, slightly paraphrasing): 'Let $\overline{AB}$ be a line segment and consider a line segment of length b where b not greater than half the length of $\overline{AB}$. We are to divide $\overline{AB}$ by a point Q such that $(AQ)(QB) = b^2$.' (Eves does not distinguish between a line segment $\overline{AB}$ and its length (AB), but we are more conscientious in these notes.) The proof is as follows (see Figure 22 below). Bisect $\overline{AB}$ at point P. Construct $\overline{PE}$ perpendicular to $\overline{AB}$ at point P. Construct a circle with E as its center and PB as its radius. Let Q be the point of intersection of the circle with $\overline{AB}$, as shown in Figure 22 (each of these constructions follow from results in Book I).

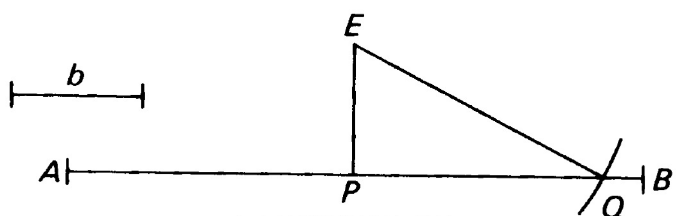


FIGURE 22

In the terminology of Proposition II.5, the "unequal segments" are segments $\overline{AQ}$ and $\overline{QB}$ (so that the "rectangle contained by the unequal segments" has area $(AQ)(QB)$). The "straight line between the points of section" is $\overline{PQ}$ (so that "square on the straight line between the points of section" has area $(PQ)^2$). The "square on the half" is the square on $\overline{AP}$ or on $\overline{PB}$, so that it has area $(PB)^2$. Proposition II.5 then implies that $(AQ)(QB) + (PQ)^2 = (PB)^2$. To verify this, we introduce Cartesian coordinates (X, Y) and consider the equation of a circle centered at the origin (which we take to be point E) with radius (PB), then we have $X^2 + Y^2 = (PB)^2$. With Q identified with the point $(X, Y) = (PQ, -b)$, we have $(PQ)^2 + (-b)^2 = (PB)^2$, or

$$b^2 = (PB)^2 - (PQ)^2 = ((PB) + (PQ))((PB) - (PQ))$$

$$= ((AP) + (PQ))((PB) - (PQ)) = (AQ)(QB).$$

Rearranging, we have $(AQ)(QB) + (PQ)^2 = (PB)^2$, as claimed in Proposition II.5. With the length of $\overline{AB}$ as $AB = a$ and the ("unknown") length of $\overline{AQ}$ as $AQ = x$ (so that $(QB) = a - x$), we can change $b^2 = (AQ)(QB)$ into $b^2 = x(a - x)$ or $x^2 - ax + b^2 = 0$. Notice that if the solutions of $x^2 - ax + b^2 = 0$ are $x = r$ and $x = s$, then (by the Factor Theorem) we have $x^2 - ax + b^2 = (x - r)(x - s) = x^2 - (r + s)x + (rs)$, so that $r + s = a$ and $rs = b^2$. We know that $(AQ) + (QB) = (AB) = a$ and $(A)(QB) = b^2$, we the solutions to $x^2 - ax + b^2 = 0$ must be (AQ) and (QB). Notice that in Note 3.7.B, only one solution of $x^2 - ax + b^2 = 0$ was constructed. In modern notation, we also have that $-(AQ) - (QB) = -((AQ) + (QB)) = -a$ and $(-(AQ))(-(QB)) = (AQ)(QB) = b^2$, so we similarly have that the solutions to $x^2 + ax + b^2 = 0$ are $-(AQ)$ and $-(QB)$, if one "believes in" or "accepts" negative numbers; but this is an idea hundreds of years into the future from the time of Euclid.

Note 3.7.E. Similarly, the special case of Proposition VI.29 given in Note 3.7.C can be addressed with a more elementary approach than that given in Book VI (as mentioned by both Eves and Heath). Euclid's Proposition II.6 states: "If a straight line segment $\overline{AB}$ is bisected and a straight line is added to it in a straight line, then the rectangle contained by the whole with the added straight line and the added straight line together with the square on the half equals the square on the straight line made up of the half and the added straight line." Eves again clarifies (we paraphrase): 'Let $\overline{AB}$ be a line segment and consider a line segment of length b. We are to extend $\overline{AB}$ to a point Q such that $(AQ)(BQ) = b^2$.' The proof is as follows (see Figure 23 below). Construct $\overline{BE}$ of length b perpendicular to $\overline{A}$ at point B. Bisect $\overline{AB}$ and denote its midpoint as P. Construct a circle centered at P with radius PQ. Let Q be the point of intersection of the circle with the extension of $\overline{AB}$, as shown in Figure 23 (each of these constructions follow from results in Book I).

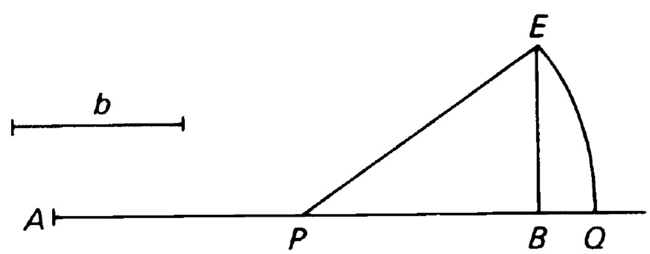


FIGURE 23

In the terminology of Proposition II.6, "the rectangle contained by the whole and the added straight line" is the rectangle with area $(AQ)(BQ)$, "the square on the half" has area $(AP)^2 = (PB)^2$, and "the square on the straight line made up of the half and the added straight line" has area $(PQ)^2$. Proposition II.6 then implies that $(AQ)(BQ) + (PB)^2 = (PQ)^2$. To verify this, we introduce Cartesian coordinates (X, Y) and consider the equation of a circle centered at the origin (where we take to be point P) with radius (PQ), then we have $X^2 + Y^2 = (PQ)^2$. With point E identified with the point $(X, Y) = (PB, BE)$, we have $(PB)^2 + (BE)^2 = (PQ)^2$ or $(PB)^2 + b^2 = (PQ)^2$ or $b^2 = (PQ)^2 - (PB)^2$. Hence $b^2 = (BE)^2 = (PQ)^2 - (PB)^2 =$

$((PQ)+(PB))(PQ)-(PB)) = ((PQ)+(AP))((PQ)-(PB)) = (AQ)(BQ)$. Therefore $b^2 = (BE)^2 = (PQ)^2-(PB)^2 = (AQ)(BQ)$, or rearranging $(AQ)(BQ)+(PB)^2 = (PQ)^2$, as claimed in Proposition II.6. Again with a as the length of $\overline{AB}$, we have $(AQ)+(-BQ) = (AB) = a$ and $(AQ)(-BQ) = -b^2$, so we have the solutions of the quadratic $x^2+ax-b^2 = 0$ are (AQ) and $-(BQ)$ (as discussed in Note 3.7.D). Similarly (changing the signs of (AQ) and $-(BQ)$), the solutions of $x^2-ax-b^2 = 0$ are $-(AQ)$ and (BQ).

Note. To summarize, we have addressed the types of quadratic equations (where $a > 0$ and $b > 0$) and produced the types of solutions as follows:

Type of quadratic	Solved in	Solutions
$x^2-ax+b^2 = 0$	Note 3.7.B	one + solution
$x^2-ax-b^2 = 0$	Note 3.7.C	one + solution
$x^2-ax+b^2 = 0$	Note 3.7.D	two + solutions
$x^2+ax+b^2 = 0$	Note 3.7.D	two − solutions
$x^2+ax-b^2 = 0$	Note 3.7.E	one +, one − solution
$x^2-ax-b^2 = 0$	Note 3.7.E	one +, one − solution

Notice that in the last four cases, we have constructed all solutions of a quadratic equation (when two real solutions exist) in the case where the coefficients a and b^2 are constructible. If we disallow negative numbers, then the negative solutions are not of interest. In fact, to avoid the use of negatives we need to rearrange the quadratics equations as:

Quadratic equation	Rearranged equation
$x^2 - ax + b^2 = 0$	$x^2 + b^2 = ax$
$x^2 + ax + b^2 = 0$	$x^2 + ax + b^2 = 0$
$x^2 + ax - b^2 = 0$	$x^2 + ax = b^2$
$x^2 - ax - b^2 = 0$	$x^2 = ax + b^2$

We'll see this need to rearrange equations to avoid negative coefficients again when we consider cubic equations and Girolamo Cardano's *Ars magna* of 1545 in Section 8.8. Cubic and Quartic Equations.

Note 3.7.F. In terms of compass and straight edge constructions, we show in Introduction to Modern Algebra 2 (MATH 4137/5137) that if $c \geq 0$ is constructible the $\sqrt{c}$ is constructible. The algebraic proof is given in my online notes for this class on Section VI.32. Geometric Constructions (see Corollary 32.8). A more geometric proof of this (using the Pythagorean Theorem) is given in the video supplement to those notes. My supplemental presentation "Compass and Straight Edge Constructions" is available as a YouTube video(accessed 3/10/2023) and a PowerPoint presentation with a written transcript (see Theorem 32.6$'$). The constructible numbers (that is, those real numbers c for which one can use a compass and straight edge to construct a line segment of length c) are classified in Theorem 32.6 of Section VI.32. Geometric Constructions as follows:

> **Theorem 32.6.** The field of constructible real numbers consists *precisely* of all real numbers that we can obtain from $\mathbb{Q}$ by taking square roots of positive numbers a finite number of times and applying a finite number of field operations.

The "field operations" mentioned here are addition and multiplication (which, using inverses, includes subtraction and division, but subtraction and division are not legitimate field operations!). The operations and the taking of square roots can be iterated so that, for example, $15 \times \sqrt{\dfrac{1+\sqrt{3}}{\sqrt{22/7 - \sqrt[4]{7/3}}}}$ is constructible. Examples of a non-constructible numbers include $\sqrt[3]{2}$, $\sqrt[5]{3/2}$, and all transcendental numbers such as π and e. The history of constructible numbers, algebraic numbers, and transcendental numbers is given in Section 14.2. Impossibility of Solving the Three Famous Problems with Euclidean Tools. These ideas also play a role in the next chapter.

Note. Since this chapter is about "Pythagorean Mathematics," we look for connections of these ideas (which we found in Euclid's *Elements*) to the Pythagoreans. We rely again on Thomas Heath's *A History of Greek Mathematics, Volume I: From Thales to Euclid* (Oxford University Press, 1921) which states (see Page 153):

> "[Some of Euclid's Book II results] were also employed by the Pythagoreans for the specific purpose of proving the property of 'side-; and 'diameter-' numbers. . . . The *geometrical algebra*. . . as we find it in Euclid Books I and II was Pythagorean. It was of course confined to problems not involving expressions above the second degree. Subject to this, it was an effective substitute for modern algebra."

Revised: 4/30/2023