3.8. Transformation of Areas

Note. In this section, we consider the transforming of one object (a triangle or a "rectilineal figure") into another object (a parallelogram or square) of the same area.

Note. In the following results from the *Elements*, a "rectilineal angle" is simply an angle between two lines or line segements. A "rectilineal figure" is a region in the plane whose border consists of line segments. The expression "equal to a given..." means that the area of the constructed

Proposition I.42. To construct, in a given rectilineal angle, a parallelogram equal to a given triangle.

Proposition I.44. To a given straight line to apply, in a given rectilineal angle, a parallelogram equal to a given triangle.

Proposition I.45. To construct, in a given rectilineal angle, or parallelogram equal to a given rectilineal figure.

Proposition II.14. To construct a square equal to a given rectilineal figure.

In the propositions of Book I, a parallelogram is to be constructed, given an angle of the parallelogram and given a desired area (either given by the area of a triangle or a rectilineal figure). Proposition II.14 is similar, but the parallelogram is required to be a square. To distinguish between the claims of Propositions I.42 and I.44, we refer to the illustrations given in David Joyce's online *Elements* webpage for Proposition I.42 and Proposition I.44.

Note. Eves declares that "[t]he Pythagoreans were interested in transforming an area of one rectilineal shape into another rectilineal shape." See page 90 of Eves. For a reference, we turn to Thomas Heath's *The Thirteen Books of Euclid's Elements, Volume I* (Cambridge University Press, 1908). This source quotes Plutarch's (46 CE–119 CE) *Non posse suaviter vivi secundum Epicurum* (c. 11), which is available online on the Persues Digital Library (accessed 3/10/2023). It states: "Pythagoras sacrificed an ox on the strength of his proposition as Apollodotus says... whether it was the theorem of the hypotenuse viz. that the square on it is equal to the squares on the sides containing the right angle, or the problem about the *application of an area*." See page 343 of Heath's *Euclid*, where he is commenting on Proposition I.44.

Note. To give an example of constructing a triangle with an area equal to that of a given rectilineal figure, consider the figure below based on Eves Figure 24. We start with the polygon $ABCDE$ (top left in the figure). Construct BR parallel to AC where R is the point of intersection of BR and DC (top center). Since the triangles ABC and ARC have a common base AC and equal altitudes on this common base, the triangles ABC and ARC have the same areas (top center and top right). Therefore polygons $ABCDE$ and $ARDE$ have the same areas. Progress has been made in the sense that $ABCDE$ has 5 sides, but $ARDE$ only has 4 sides. So this process can be iterated until we produce a triangle. Next, construct ES parallel to AD where S is the point of intersection of ES and DC (bottom left). Since the triangles ADE and ADS have a common base AD and equal altitudes on this common base, the triangles ADE and ADS have the same areas (bottom left and bottom center). Therefore polygon $ABCDE$ triangle ARS have the same areas (top left and bottom right).