3.9. The Regular Solids

Note. In this section we describe the five Platonic solids and reference Euclid's *Elements* on their existence and uniqueness. We give references addressing the Pythagorean's knowledge of these solids.

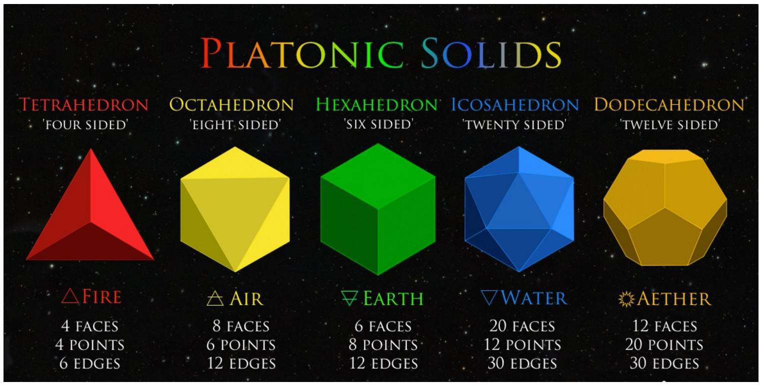
Definition. A polyhedron is *regular* if its faces are congruent regular polygons and if its polyhedral angles are congruent.

Note 3.9.A. There are exactly five regular polyhedra: tetrahedron, hexahedron (i.e., cube), octahedron, dodecahedron, and icosahedron. Each of these are constructed by inscribe them in a sphere in Euclid's Book XIII. Proposition XIII.13 gives the construction of a tetrahedron (called a "pyramid" in the *Elements*), Proposition XIII.14 gives the construction of an octahedron, Proposition XIII.15 gives the construction of a cube, Proposition XIII.16 gives the construction of an icosahedron, and Proposition XIII.17 gives the construction of a dodecahedron. That is, each of the five regular polyhedra exists. In Proposition XIII.18 (the last numbered result in the *Elements*) concerns the lengths of the sides of each regular solid in terms of the radius of the sphere in which it is inscribed. The last result in the *Elements* (other than a lemma used in the proof of the last result) states (see Heath's *Euclid: The Thirteen Books of the Elements, Volume 3*, page 507): "I say next that *no other figure, besides the said five figures, can be constructed which is contained by equilateral and equiangular figures equal to one another*." That is, the given polyhedra are the only regular polyhedra. These five regular polyhedra are usually called the *Platonic solids*.

Note 3.9.B. In Carl Sagan's *COSMOS*, Random House 1980 (Appendix 1 of which gives a proof that $\sqrt{2}$ is irrational, and Appendix 2 gives a proof based on Euler's Formula that there are at most five regular polyhedra) the mystical view the Pythagoreans held concerning the five regular solids is addressed (see pages 184 and 185):

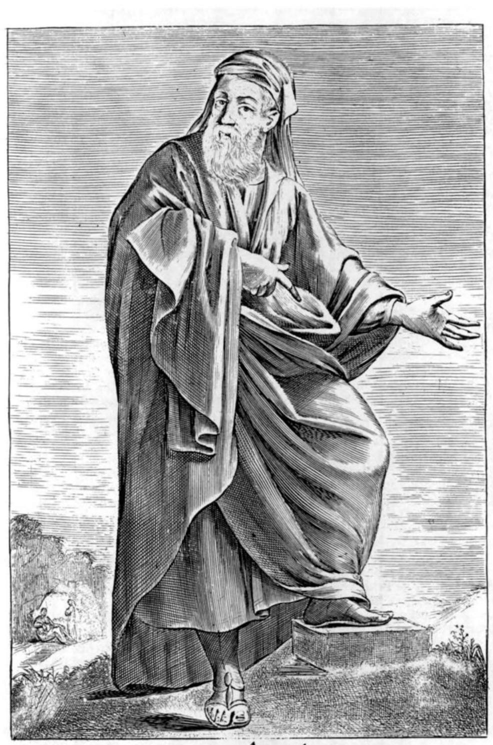
> "For some reason, knowledge of a solid called the dodecahedron having twelve pentagons as sides seemed to them dangerous. It was mystically associated with the Cosmos. The other four regular solids were identified, somehow, with the four "elements" then imagined to constitute the world: earth, fire, air and water. The fifth regular solid must then, they thought, correspond to some fifth element that could only be the substance of the heavenly bodies. (This notion of a fifth essence is the origin of our word quintessence.) Ordinary people were to be kept ignorant of the dodecahedron."

The Hellenic Faith website (accessed 3/3/2023) gives the following image illustrating the solids and their connection to the ancient "elements."



Note. The argument that there are at most five regular polyhedra based on Euler's formula is also given in Foundations and Structure of Mathematics 1 (MATH 5025). See my online notes for this class on The Platonic Solids.

Note 3.9.C. As we quoted earlier in these notes in Supplement. Proclus's Commentary on Eudemus' *History of Geometry*: "He [Pythagoras] it was who discovered the doctrine of proportionals and the structure of the cosmic figures [the five regular solids]." (See the first paragraph of the quoted Eudemus history.) According to the Empedocles of Acragas MacTutor biography page, Empedocles (492 BCE–432 BCE) proposed that all matter is composed of four elements, fire, air, water, and earth. This idea was was adopted by Plato and Aristotle and widely spread. This idea, like the ideas of the Pythagoreans, tried to explain the large, complex world in terms of a small number of underlying properties. The empirical science we use today still looks for simple mathematical explanations of complex processes in the universe.



Empedocles.

Engraving of Empedocles from Thomas Stanley's 1655 *The History of Philosophy*; from the Wikipedia page on Empedocles (accessed 3/4/2023)

Note 3.9.D. In Thomas Heath's *A History of Greek Mathematics, Volume I: From Thales to Euclid* (Oxford University Press, 1921; see pages 158) he attributes the regular solids to the Pythagoreans:

> "...it would appear that Plato in the *Timaeus* is the earliest authority for the allocation, and it may very well be due to Plato himself (were not the solids called the 'Platonic figures'?), although put into the mouth of a Pythagorean [Timaeus]."

Plato even describes gives constructions of the five regular solids in *Timaeus* by connecting equilateral triangles, squares, and pentagons at a point in such a way to complete the solid angles. Heath argues that the Pythagoreans could have discovered the regular solids in a similar way (see his page 159). Since four of the five regular solids were associated with Empedocles' four elements, is had been argued that the Pythagoreans were unaware of the dodecahedron (or maybe even of the icosahedron). Heath quotes in scholium No. 1 to Euclid's Book XIII as (see Heath's page 162): "the five so-called Platonic figures, which, however, do not belong to Plato, three of the five being due to the Pythagoreans, namely the cube, the pyramid, and the dodecahedron, while the octahedron and icosahedron are due to Theaetetus." Heath proposes that this impression could be held due to the fact that Theaetetus wrote extensively about the last two solids. To construct the dodacahedron, a pentagon is needed. We mentioned Iambilucus' (circa 250 CE–circa 330 CE) lengthy *The Life of Pythagoras* in Section 3.2. Pythagoras and the Pythagoreans and we mentioned the Pythagorean Hippasus in Section 3.5. Discovery of Irrational Magnitudes. In Iambilucus' biography, Hippasus is linked to the dodecahedron (as quoted by Heath on his page 160): "...owing to [Hippasus] being the first to publish and write down the (construction of the) sphere with...twelve pentagons, perished by shipwreck [was drowned?] for his impiety, but received credit for the discovery, whereas it really belonged to [Pythagoras]. . . .”

Plato and Aristotle from Raphael’s 1510 *The Academy*. Plato is holding a copy of his book *Timaeus*. This is from the Wikipedia page on Plato (accessed 3/4/2023)

Note 3.9.E. Eves mentions on page 92 that Johann Kepler (December 27, 1571–November 15, 1630), who Eves classifies as “master astronomer, mathematician, and numerologist,” gave some explanation of Timaeus’ ideas concerning the elements and the Platonic solids. He also read something into the fact that a dodecahedron has 12 sides and the zodiac in the heavens/universe has 12 signs. A more famous idea of Kepler’s is that the Platonic solids could be used to explain the orbits of the then-known six planets. By considering spheres that are inscribed in or circumscribed around the five platonic solids (this is how six orbits are determined by five solids; we see again a numerological connection), he attempted to model the movements of the planets in the night sky. His model did not match the observed date (much of which was made by Tycho Brahe, December 14, 1546–October 24, 1601; all of Brahe's observations were naked-eye and precede the invention of the telescope). Of course Kepler was ultimately successful in modeling the movement of the planets when he gave his three laws of planetary motion. Kepler's successes and failures are discussed in the history component of Introduction to Modern Geometry (MATH 4157/5157). See my online notes for this material on Section 5.10. The Great Discoveries of Kepler and Newton. We'll see Johann Kepler again in this class in Section 9.7. Kepler.

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