Chapter 4. Duplication, Trisection, and Quadrature

4.1. The Period from Thales to Euclid

Note. In this section we consider the political and military history of the regions with coasts on the northwestern part of the Mediterranean Sea. This includes Anatolia (modern day Turkey), Greece, and many of the islands in the Aegean sea. We cover the time period from about 1200 BCE to about 350 BCE. We also briefly consider the work done in mathematics and philosophy in the region over this time period.

Note. In the last chapter we surveyed the mathematics of Thales of Miletus (circa 624 BCE–circa 547 BCE) in in Section 3.1. Birth of Demonstrative Mathematics. We considered Pythagoras of Samos (circa 570 BCE–circa 490 BCE) and the Pythagorean school in Section 3.2. Pythagoras and the Pythagoreans and throughout the rest of Chapter 3. The limited available information on other Greek mathematicians before Euclid is given in Supplement. Proclus's Commentary on Eudemus' *History of Geometry*. Chapter 5 is devoted to Euclid (circa 325 BCE–circa 265 BCE), the *Elements*, and his other works.

Note 4.1.A. The first advanced and distinctively Greek civilization in mainland Greece was the Bronze Age Mycenaean Greek culture, with its palaces, urban organization, works of art, and writing system. It lasted roughly from 1750 BCE to

1050 BCE and ended with the Late Bronze Age collapse, after which the palatial states were replaced with the small isolated villages of the "Greek Dark Ages" (or the "Homeric Age," since Homer's *Illiad* and *Odyssey* and most of Greek mythology and religion dates from this time). Eves credits the decline of Mycenaean Greece to "the primitive Dorian tribes [who] moved southward into the Greek peninsula" displacing the former occupants into "Asia Minor [modern day Turkey] and the Ionian islands of the Aegean Sea" (Eves, page 105). However, there is no archaeological evidence of this (see the Wikipedia webpage on the History of Greece). The term "Ancient Greece" refers to the time from 1200 BCE to 600 CE, and is further subdivided into other the Greek Dark Ages, the Archaic Period (800 BCE–490 BCE), the Classical Period (490 BCE–323 BCE), the Hellenistic Period (323 BCE–146 BCE), and Roman Greece (146 BCE–324 BCE). During the Archaic Period, the Greek alphabet, early Greek literature, and the city states ("*polis*") were developed. Also during this time, Greeks spread to the shores of the Black Sea, Southern Italy (the "Magna Graecia") and Anatolia (i.e., Asia Minor, which is modern day Turkey). The central part of the western coast of Anatolia formed the region known as Ionia (which also includes some of the islands in the Aegean, including Chios and Samos). Notice that Ionia is on the Aegean Sea and not on the Ionian Sea, as we might expect! It is in Ionia that the Ionian school of Pre-Socratic philosophy started in the sixth century BCE. The school is associated with Thales of Miletus (circa 624 BCE–circa 547 BCE), Anaximander (circa 610 BCE–circa 546 BCE)), and Anaxagoras (499 BCE–428 BCE). As we observed in Section 3.1. Birth of Demonstrative Mathematics, demonstrative geometry began with Thales (Eves, page 72), but he is also regarded as the earliest Western philosopher. Anaximan-

der belonged to the "Milesian school" (i.e., in Miletus), along with Thales. He may have had Pythagoras as a student (this is claimed in *Life of Pythagoras* by Porphyry of Tyre [233 CE–309 CE]). He has interests in astronomy, physics, geography, and geometry. Anaxagoras was "the last eminent member of the Ionian school" (Eves, page 106) and, as we'll mention in Section 4.7. Quadrature of the Circle, possibly the first to consider a compass and straight edge construction of squaring the circle. The sources for this note are the Wikipedia webpages on the History of Greece, Mycenaean Greece, the Ionian School, and Anaximander (each was accessed 8/15/2023).

The Eastern Mediterranean in classic times (based on Eves' page 107 and a map on the Wikipedia Commons webpage; accessed 8/15/2023)

Note 4.1.B. The Classical Period (490 BCE–323 BCE) includes the development of much of the early defining politics, artistic thought (architecture, sculpture), scientific thought, theater, literature and philosophy of that would determine Western civilization and later influence the Roman Empire. This period included the philosophers Socrates (circa 470 BCE–399 BCE), Plato (circa 425 BCE–circa 348 BCE), and Aristotle (384 BCE–322 BCE), the Parthenon in Athens, a temple dedicated to the goddess Athena, was built (447 BCE–432 BCE), and the Artemision Bronze statue (sometimes called the "God from the Sea"; it is possibly Poseidon or Zeus) was created.

Images from the Rick Steves website (host of the public television show *Rick Steves' Europe*) of the Artemision Bronze in the National Archaeological Museum of Athens (left), and the Wikipedia webpage on the Parthenon (both accessed 8/16/2023).

Note 4.1.C. The Classical Period saw two major wars: the Persian War (or the Greco-Persian War; 499 BCE–449 BCE) and the Peloponnesian War (431 BCE–404 BCE). First, the Persian Empire (modern day Iran) expanded west into Anatolia and conquered the Greek colonies there and the Ionian cities. This caused a number of Greek philosophers, like Pythagoras and Xeonophanes, to leave the area for

southern Italy. Schools of philosophy and mathematics were developed there in Crotona (by Pythagoras) and Elea (by Xenophanes, Zeno, and Pharmenides). See Eves, page 105. The Ionians revolted against the Persian invaders in 499 BCE, but were defeated in 494 BCE. Athens sent aid to the Ionians in the conflict, which inspired the Persian King Darius to attack mainland Greece in 492 BCE in the first invasion of Greece. However, his fleet of around 1200 Persian ships were lost in a storm and the invasion failed. In 490 BCE another fleet was sent to take Athens, but they were defeated at the Battle of Marathon (this battle gives its name to 26 mile foot race; the Greek messenger Philippides saw a Persian vessel headed towards Athens and he ran the 26 miles to Athens to report what he saw, after which he dropped dead...so the legend says). In 480 BCE Xerxes (the son of Darius) attempted another land and sea assault in the second invasion of Greece. The Athenians won at sea in the Battle of Salamis, but lost the land battle in the Battle of Thermopylae when they attempted to stop the advancing Persian army at a narrow pass (famous for the 300 Spartans who fought to the death in the battle). In 479 BCE, the Athenian fleet won the Battle of Mycale, resulting in the Persians being chased from the Aegean Sea. Afterwards in 478 BCE, Athens organized the 200 or 300 of the island states (and some of the mainland ones, excluding Sparta) into an alliance called the Delian League. This solidarity lead to several decades of peace. Athens became a center for democratic and intellectual development. It attracted mathematicians from throughout the Greek world, including Anaxagoras, Zeno, and Hippocrates of Chios (Eves, page 106). We'll further explore Zeno's work in Section 11.2. Zeno's Paradoxes. The source for this not, in addition to Eves, is the Wikipedia webpage on Classical Greece (accessed 8/16/2023).

Note 4.1.D. Shortly after the creation of the the Delian League, Athens began to use its funds for their own purposes. This caused conflict with the other members of the league. In 431 BCE this conflict, combined with the dominance of Athens over Sparta, lead to the outbreak of the Peloponnesian War. The first phase of the war began when Athenian general Pericles recommended fighting a defensive war, avoiding battles against the stronger forces of the Spartans. Athens would depend on its navy to import everything it needed and simply outlast Sparta. This resulted in several sieges of Athens, and in 430 BCE a plague broke out that killed one quarter of the Athenians (including Pericles). With new leaders, Athens went on the offensive. Both sides experienced victories and defeats. Compromises were made and the Peace of Nicias was signed in 421 BCE.

Greece and the Peloponnesian Peninsula (in blue), based on an image from the Wikipedia webpage on the Peloponnesian Peninsula, "Peloponnese" (left) and the Wikipedia webpage on the Peloponnesian League (both accessed 8/16/2023)

The peace lasted until the second stage of the Peloponnesian War began in 415

BCE. Athens offered support to Segesta in Sicily against an attack by Syracuse (a Spartan ally, also in Sicily). Sicily offered support to Syracuse, and Athens was defeated in this campaign. In 407 BCE, Spartan general Lysander fortified Sparta's naval forces and enjoyed several victories. In 405 BCE at the Battle of Aegospotami, Lysander's forces almost destroyed the Athenian fleet. Athens surrendered the next year in 404 BCE, ending the war. This note is based on the Wikipedia webpages on the History of Greece and the Delian League (accessed 8/16/2023).

Note. During the Persian War and the Peloponnesian War, little progress was made in geometry at Athens, and advances during this time was made in the Magna Graecia (settlements of Greeks on the southern Italian coast). A new Pythagorean school was founded at Tarantum (today, "Taranto" on the Mediterranean near the heel of the Italian boot) under the influence of Greek mathematician and philosopher Archytas (circa 428 BCE–circa 350 BCE). See Eves, pages 106.

Note 4.1.E. Following the Peloponnesian War, Athens was only a minor political power but it regained its cultural leadership. Plato (427 BCE–347 BCE) was born near Athens in the year of the plague mentioned above. He studied philosophy under Socrates (circa 470 BCE–399 BCE), and then traveled and studied. He studied mathematics under Theodorus of Cyrene (465 BCE–398 BCE; see the map of the Eastern Mediterranean above for the location of Cyrene), and became friends with Archytas of Tarantum (circa 428 BCE–circa 350 BCE). In 387 BCE, Plato founded his Academy, which he presided over until his death at the age of 80 in 347 BCE.

The Academy was located outside the city walls of Athens, near a grove of olive trees dedicated to the goddess Athena. The location was rediscovered in the 20th century. It is located about one mile north of Athens' Dipylon gates. It is currently an archaeological site that can be visited. The Academy did not have a specific curriculum nor did it (at least during Plato's time) push any particular doctrine. Plato and his associates posed problems to be studied and solved by others. As opposed to lectures, most of the learning was through dialogue between those of different viewpoints who tried to reach an agreeable reasoned conclusion.

The School at Athens by Raphael 1483–April 6, 1520), located in the Vatican Museums. This image is from the Wikipedia webpage on the Platonic Academy

This note is based on Eves' page 106 and the Wikipedia webpage on the Platonic Academy (accessed 8/16/2023).

Note 4.1.F. Eves states on pages 126 and 127:

> "Almost all the important mathematical work of the fourth century B.C. was done by friends or pupils of Plato, making his Academy the link between the mathematics of the earlier Pythagoreans and that of the later, long-lived school of mathematics at Alexandria. Plato's influence on mathematics was not due to any mathematical discoveries he made, but rather to his enthusiastic conviction that the study of mathematics furnished the finest training for the mind and, hence, was essential for the cultivation of philosophers and those who should govern his ideal state. This explains the renowned motto over the door of his Academy: *Let no one unversed in geometry enter here.* ... Some see in certain of Plato's dialogues what may be considered the first serious attempt at a philosophy of mathematics."

A close-up of Raphael's *The School at Athens* showing Plato. Image from Lapham's Quarterly website (accessed 8/17/2023).

The philosophy of mathematics that Eves mentions is called *Platonism*. For an

explanation of this, we turn to Mario Livio's *Is God a Mathematician?* (Simon & Schuster, 2009). On page 34 of this work Livio states:

> "...Plato stresses, mathematical truths refer not to circles, triangles, and squares that can be drawn on a piece of papyrus, or marked with a stick in the sand, but to abstract objects that dwell in an ideal world that is the home of true forms and perfections. This Platonic world of mathematical forms is distinct from the physical world, and it is in this first world that mathematical propositions, such as the Pythagorean theorem, hold true. The right triangle we might draw on paper is but an imperfect copy—an approximation—of the true, abstract triangle."

That is, the objects of mathematics have some type of innate existence. They are out there, waiting for us to discover them. In contrast to Platonism, is *formalism*. The Wikipedia webpage on Formalism in mathematics (accessed 8/17/2023) addresses formalism as follows:

> "According to formalism, the truths expressed in logic and mathematics are not about numbers, sets, or triangles or any other coextensive subject matter—in fact, they aren't 'about' anything at all. Rather, mathematical statements are syntactic forms whose shapes and locations have no meaning unless they are given an interpretation (or semantics)."

That is, the objects of mathematics are simply (nonexistent) objects that have no properties other than those given in a axiomatic system (and those derived through logic from those axioms). A famous quote (possibly apocryphal) concerning the mathematical objects of geometry by David Hilbert (January 23, 1862–February 14, 1943) is: "One must be able to say at all time—instead of points, straight

lines, and planes—tables, chairs, and beer mugs." (Ivor Grattan-Guiness on page 208 of his book *The Search for Mathematical Roots, 1870–1940* (Princeton University Press, 2000), gives the reference for a variant of this quote as pages 402 and 403 of O. Blumenthal's (1876–1944) *Lebensgeschichte* ["Biography"] (self published, 1935).) What the quote means is that a "line" is nothing more than a conceptual object that has the properties given to it by the axioms of Euclidean (or one of the non-Euclidean) geometry. It isn't *really* anything! We may as well call it a table (though this analogy isn't precise, since a table *really is* something; we may as well call it something meaningless like a "blurgum"). This reduces the act of doing mathematics to the formal manipulations of symbols according to the properties given to the objects by the axioms and the implications that formal logical manipulation of the symbols produces. Your humble instructor prefers formalism to Platonism (thus putting my in the minority among working mathematicians, so I understand). We'll see more of this in Section 15.8. Philosophies of Mathematics; see also my supplemental notes for Great Ideas in Science 1 (BIOL 3018) on Supplement. Introduction to Math Philosophy and Meaning.

Note 4.1.G. We now consider several others from the western Mediterranean who made mathematical contributions before Euclid. We first introduced Eudoxus of Cnidus (408 BCE–355 BCE) in connection with his theory of proportion (which appears in Book V of the *Elements*) in Section 3.5. Discovery of Irrational Magnitudes; see Note 3.5.E. More details on his theory of proportions is in Section 5.4. Content of the "Elements" in Notes 5.4.J and 5.4.K. Eudoxus studied under both Archytas (circa 428 BCE–circa 350 BCE) and Plato (427 BCE–347 BCE), and he founded a

school at Cyzicus in northwestern Anatolia (in the Balikesir Province of modern day Turkey). We'll consider Eudoxus' work in Section 11.3. Eudoxus' Method of Exhaustion. Menaechmus (circa 380 BCE–circa 320 BCE) was an associate of Plato and a pupil of Eudoxus. He is credited with the invention of the conic sections, which he then used to double the cube. See Note 4.5.B in Section 4.5. Duplication of the Cube and my online notes for the history component of Introduction to Modern Geometry (MATH 4157/5157) on Chapter 3. Conic Sections (Note 3.A). Dinostratus (circa 390 BCE–circa 320 BCE) was a brother of Menaechmus and a pupil of Plato. He is presumed to be the one who used the quadratrix of Hippias to square the circle (see Note 4.7.B of Section 4.7. Quadrature of the Circle). Theaetetus of Athens (circa 417 BCE–circa 369 BCE) was a student of Theodorus and is probably the one to whom much of the content of Euclid's Books X and XIII of the *Elements* should be credited. Theaetetus' contributions to Book X concern irrationals, commensurables, and uncommensurables (see Note 5.4.0 of Section 5.4. Content of the "Elements"). In connection to Book XIII (on the Platonic solids), the octahedron and icosahedron are due to Theaetetus (see Note 3.9.D of Section 3.9. The Regular Solids). The Greek philosopher Aristotle (384 BCE–322 BCE) was not a mathematician, but he systematized deductive logic, which is the main tool of mathematicians. His approach to logic is known variously as Aristotelian logic, term logic, traditional logic, or syllogistic logic. Aristotle published six works which make of the *Organon*: I. Categories, II. On Interpretation, III. Prior Analytics, IV. Posterior Analytics (which shows a familiarity of the mathematical method), V. Topics, and VI. On Sophistical Refutations (a copy of these works in English is available at Archive.org; accessed 8/17/2023). See my online notes for Mathemat-

ical Reasoning (MATH 3000) on Section 1.4. Proofs: Structures and Strategies for a little more on Aristotelian logic; for a modern approach (based on truth tables), see Section 1.2. Logical Connectives and Truth Tables, Section 1.3. Conditional Statements, and Section 1.5. Logical Equivalence.

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