4.4. The Euclidean Tools

Note. In this section we give the properties of the compass and straight edge, as endowed to them by the first postulates of Euclid's *Elements*. We consider both a "Euclidean compass" and a "modern compass." Some general history of constructions is given, and we explore the implications of using *only* a compass and straight edge in geometry.

Note. The straight edge allows us to construct ("draw") a straight line of any length through any two (distinct) given points. This is built into Euclid's postulates of Book I in Postulate 1 and Postulate 2:

Postulate 1. To draw a straight line from any point to any point.

Postulate 2. To produce a finite straight line continuously in a straight line.

Postulate 1 tells us that *given* any two points A and B, the line segment $\overline{AB}$ can be constructed (an assumption of uniqueness should be added). Postulate 2 tells us that any line segment can be extended indefinitely. A subtlety of Postulate 1 is that the two points A and B must already be determined (for example, a point is determined by the intersection of two nonparallel lines); that is, the points must be constructed before the line or line segment can be created.

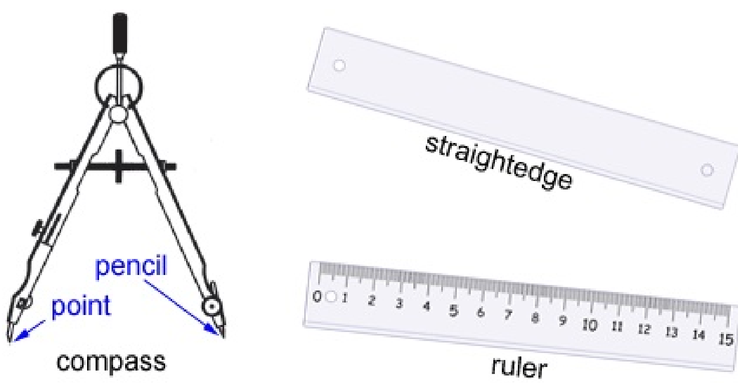
Note. The compass allows us to construct ("draw") a circle with any given point as center and passing through any given second point (this is Eves' statement of the use of the compass; see page 110). This is built into Euclid's third postulate of

Book I:

Postulate 3. To describe a circle with any center [i.e., given point] and distance [i.e., radius as determined by a second given point].

Again, we must already have the center of the circle determined (or "constructed") and we must already have the radius constructed in the sense that we have constructed a line segment with length equal to the desired radius (or the second point must already be constructed). Uniqueness should be assumed in Postulate 3 also.

Note/Definition. A compass and straight edge used in a way as to satisfy Postulates 1, 2, and 3, make up the *Euclidean tools*. The straight edge is unmarked and only used to create lines or line segments through already-constructed points; it is not used like a *ruler* to measure distance.

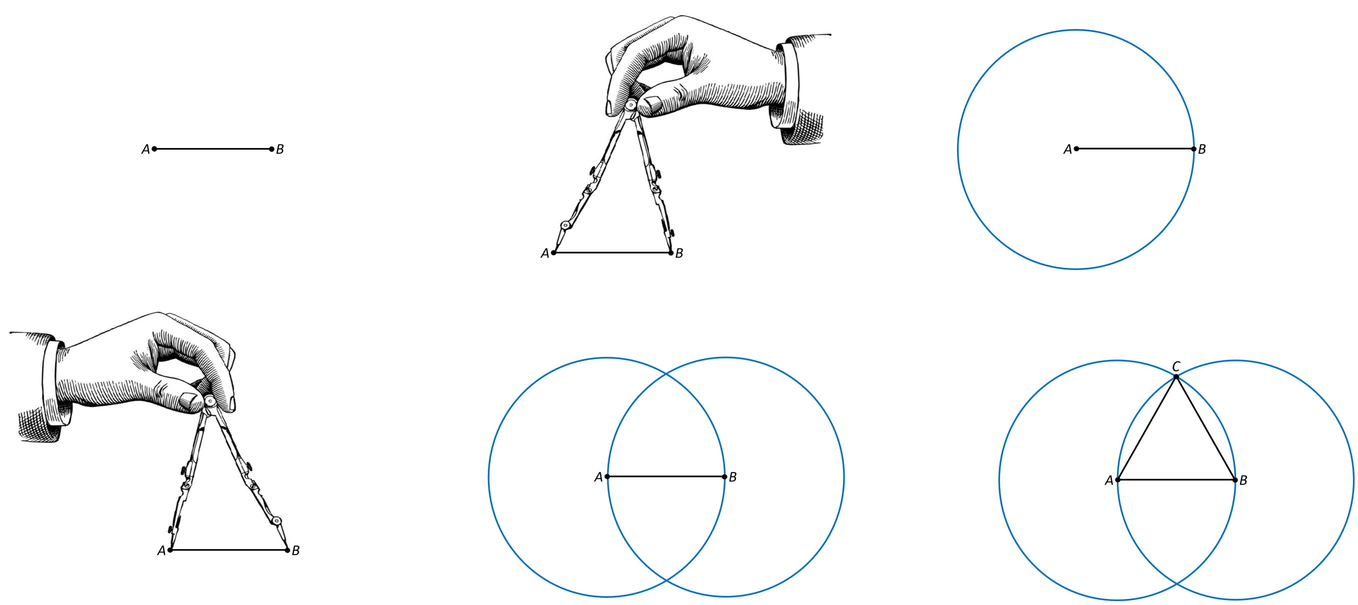


From the Geometric Constructions webpage of MATH.net (accessed 3/11/2023)

A Euclidean compass cannot be used to (directly) transfer a given distance AB to the given center C; it "may be supposed to collapse if either leg is lifted from the paper" (as Eves says on page 110). The points of a *modern compass* can be places at given points A and B, and then "locked" into place so that the distance AB can be translated to point C. So a modern compass has more abilities than a Euclidean compass. However, any construction that can be performed with a modern compass and straight edge can also be performed with a Euclidean compass and straight edge. This can be shown using Proposition I.2 ("To place at a given point (as an extremity [i.e., as an endpoint]) a straight line equal to a given straight line"), as is to be done in Problem 4.1.

Note. As observed in Supplement. Proclus's Commentary on Eudemus History of Geometry, Thomas Heath in his *A History of Greek Mathematics, Volume I. From Thales to Euclid* (Clarendon Press, Oxford, 1921) speculates that Oenopides made compass and straight-edge constructions central to geometry: "It may therefore be that Oenopides's significance lay in improvements of method from the point of view of theory; he may, for example, have been the first to lay down the restriction of the means permissible in constructions to the ruler and compasses which became a canon of Greek geometry for all 'plane' constructions, i.e. for all problems involving the equivalent of the solution of algebraical equations of degree not higher than the second." See Heath's pages 175 and 176. Although the first three postulates of the *Elements* clearly do lay out an approach motivated by compass and straight edge constructions, Heath also comments in his translation of Euclid's *Elements*, Volume 1 (see his page 124): "There is of course no foundation for the idea, which has found its way into many text-books, that 'the object of the postulates is to declare that the *only instruments* [emphasis added] of the use of which are permitted in geometry are the *rule* and *compass*.' "

Note 4.4.A. As a quick illustration of a compass and straight edge proof, we consider Euclid's Proposition I.1: "On a given finite straight line [segment] to construct an equilateral triangle." We start with line segment $\overline{AB}$ (see the figure below, upper left). First, we use the compass to draw a circle centered at A with radius AB (so that the stationary point of the compass is placed at A and the drawing point of the compass is placed at B; see the figure, upper center and upper right). Second, we use the compass to draw a circle centered at B with radius AB (see the figure, lower left and lower center). The resulting circles intersect at two points. Label one of the points as C. Use the straight edge to introduce line segments $\overline{AC}$ and $\overline{BC}$ (see the figure, lower right). Since distances AB, AC, and BC are all radii of circles of radius AB, then $AB = AC = BC$ and the triangle ABC is an equilateral triangle, as claimed.



Note 4.4.B. As we will discuss in the next three sections, if we restrict our tools to a compass and straight edge, then there are certain constructions that cannot be performed (in particular, the "Three Famous Problems"). In terms of numbers (i.e., distances), we start with some line segment which we use to designate the unit length. Then, based on that unit length, other length segments can be constructed. For example, if line segment $\overline{AB}$ in our proof of Proposition I.1 is of length 1 and we construct the line segment from point C to the midpoint of $\overline{AB}$ (line segments are bisected in Proposition I.10), then we have constructed a line segment of length $\sqrt{3}/2$ (by the Pythagorean Theorem). We have already observed (in Section 3.7. Geometric Solution of Quadratic Equations; see Note 3.7.F) that the constructible numbers are precisely the real numbers that can be obtain from $\mathbb{Q}$ by taking square roots of positive numbers a finite number of times and applying a finite number of field operations. This is Theorem 32.6 in my online notes for Introduction to Modern Algebra 2 (MATH 4137/5137) on Section VI.32. Geometric Constructions. It is this modern algebra result from the 1800s that will give the unsolvability of the three famous problems. A very interesting geometry paper by Wendell Strong titled "Is Continuity of Space Necessary to Euclid's Geometry?" was published in the *Bulletin of the American Mathematical Society*, **4**(9), 443–448 (June 1898), and is available on the AMS webpage In this paper, Strong considers (see his page 444) "the least space in which the constructions of Euclid are possible; it contains the points which can be obtained by a finite number of these constructions and no others." He effectively introduces Cartesian coordinates to the plane $\mathbb{R}^2$, and then eliminates all point with at least one coordinate a nonconstructible number. Due to the nature of constructible numbers in terms of the taking of square roots, Strong uses the term "quadratic number" instead of "constructible number." He then defines what he calls *quadratic space* consisting of all points in $\mathbb{R}^2$ such that the first and second coordinates are quadratic numbers. Strong considers a three dimensional space, but our purposes are served by a two dimensional "quadratic plane." He claims the following two results:

Theorem 1. Any two points of the quadratic space are at a quadratic distance from each other.

Theorem 2. A point of continuous space at quadratic distances from three points of the quadratic space is a point of the quadratic space.

The surprising result of all of this, is that continuity is not needed to do the com--pass and straight edge geometry of Euclid! Strong's space is shot full of holes! Not to go too far astray, but there are only countably many quadratic numbers (they are special cases of algebraic number—an algebraic real number is a real number that is the root of some polynomial with integer coefficients—and there are only countably many algebraic numbers). There are uncountably many real numbers. The implication of these facts is that there are only countably many points in in the quadratic plane (or in quadratic space), but there are uncountably many points in $\mathbb{R}^2$. Think of it as the quadratic plane has infinitely many points, but it's a small infinity! The Cartesian plane $\mathbb{R}^2$ has in infinite number of points as well, but it is a bigger kind of infinity! Cardinalities of infinite sets are possibly discussed in Mathematical Reasoning (MATH 3000) and definitely in Analysis 1 (MATH 4127/5127). See my online notes for these classes, respectively, on Section 4.3. Countable and Uncountable Sets and Section 1.3. The Completeness Axiom. The quadratic plane and quadratic space have an uncountable infinities of holes in them!