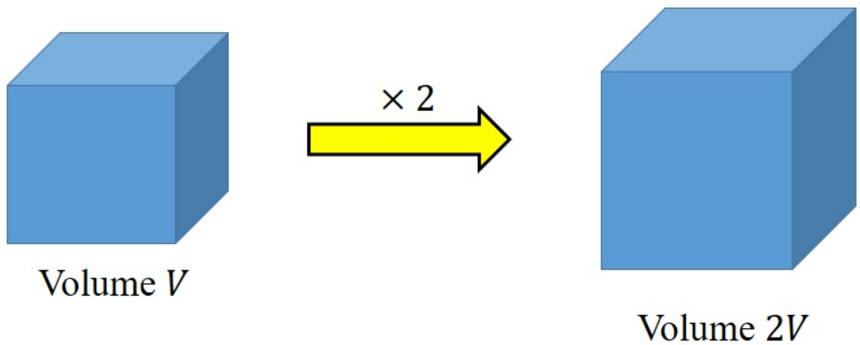
4.5. Duplication of the Cube

Note. In this section we discuss the classical compass and straight edge problem of constructing a cube of twice the volume of a given cube. This problem is called "duplication of the cube" or (better) "doubling the cube." This is equivalent to constructing the number $\sqrt[3]{2}$. We give some history of the problem and discuss constructions that involve more than just a compass and straight edge.



Note. Let's start with the punch-line. Recall that the constructible numbers are precisely the real numbers that can be obtain from $\mathbb{Q}$ by taking square roots of positive numbers a finite number of times and applying a finite number of field operations (see Note 3.7.F of Section 3.7. Geometric Solution of Quadratic Equations). This is Theorem 32.6 in my online notes for Introduction to Modern Algebra 2 (MATH 4137/5137) on Section VI.32. Geometric Constructions. So, the number $\sqrt[3]{2}$ is not constructible and the cube cannot be doubled with a compass and straight edge! This was not known until the development of modern algebra in the 1800s, so there is a rich history to the problem (the history of algebraic solution by Pierre Wantzel in 1837 is given in Section 14.2. Impossibility of Solving the Three Famous Problems with Euclidean Tools). Additional tools (other than a compass and straight edge) have been proposed to solve the construction problem.

Note. Thomas Heath in his *A History of Greek Mathematics, Volume I: From Thales to Euclid* (Oxford University Press, 1921) credits Eutocius's (480 CE–540 CE) commentary on Archimedes, *On the Sphere and Cylinder*, II. 1, as "a precious collection of solutions of this famous problem." See Heath's page 244; Heath gives a detailed history of the problem on his pages 244–270 and this is likely the source of Eves' presentation. The story (told by an unknown ancient poet) tells of the mythical King Minos' dissatisfaction with a tomb erected to his son Glaucus. He demanded that it be made twice the size (i.e., twice the *volume*) and that this could be accomplished by doubling the linear dimensions of the tomb. Of course, doubling the linear dimensions results in increasing the volume by a factor of eight. In a purported letter from Eratosthenes of Cyrene (276 BCE–194 BCE) to Ptolemy Euergetes (the pharaoh of Egypt from 246 BCE to 222 BCE; not be confused with second century Roman astronomer and mathematician Claudius Ptolemy) it is stated: "Geometers took up the question and sought to find out how one could double a given solid while keeping the same shape; the problem took the name of 'the duplication of the cube' because they started from a cube and sought to double it." Progress was minimal until Hippocrates of addressed the problem by considering two mean proportionals (discussed below), but this simply translated the problem into one concerning proportionals. The story continues that the Delians (that is, people of Greek island of Delos) were commanded by their oracle to double a certain altar and, trying to apply Hippocrates solution, came across the same difficulties as in the original construction problem. It seems that the problem was attacked at Plato's Academy, some of the stories say ("though probably erroneously," Eves page 111) that Plato himself gave a solution. Heath also states (see his page 246): "After Hippocrates had discovered that the duplication of the cube was equivalent to finding two mean proportionals..., the problem seems to have been attached in [this] form exclusively." Heath goes on to describe solutions by Archytas (circa 400 BCE solution based on intersections not in the place, but of surfaces in space), Eudoxus, Menaechmus (two solutions), Pluto (attributed to Plato), Eratosthenes, Nicomedes, Appolonius, Heron, Philon (these three giving similar solutions), Diocles (using the cissoid), Sporus, and Pappus (these last two gave the same solution as Diocles). In these notes, we concentrate on the solutions of Hippocrates, (attributed to) Plato, and Menaechmus.

Note 4.5.A. Hippocrates of Chios (circa 470 sc bce–circa 410 BCE) addressed the problem around 440 BCE by reducing it to the construction of two mean proportionals between two given line segments of lengths s and $2s$. Stated in terms of proportions with the two mean proportions as x and y, he considers $s : x = x : y = y : 2s$. With these "proportions" written as quotients we have $\frac{s}{x} = \frac{x}{y} = \frac{y}{2s}$, or $sy = x^2$ and $y^2 = 2sx$. This then gives $y^2 = \left(\frac{x^2}{s}\right)^2 = \frac{x^4}{s^2}$ and hence $y^2 = \frac{x^4}{s^2} = 2sx$ or $x^3 = 2s^3$. Then with s as the length of a side of a given cube (that is, a cube of given volume s^3) the x is the "unknown" length of a side of a cube of twice the volume of the given cube: $x^3 = 2(s^3)$.

Note 4.5.B. Menaechmus (circa 380 BCE–circe 320 BCE) gave two solutions to Hippocrates' two mean proportionals problem. His results are described by Eutocius in his commentary on Archimedes' *On the Sphere and Circle*. Both solutions involve finding a point of intersection of two conic sections. In one solution he considers the solution of two parabolas, and in the other he consideres the intersection of a parabola and a hyperbola. Based on these solutions "it is inferred that Menaechmus was the discoverer of the conic sections." See Heath, page 251. The first solution follows from the two mean proportionals problem of Note 4.5.A: $\frac{s}{x} = \frac{x}{y} = \frac{y}{2s}$. This implies $y = \frac{x^2}{s}$ and $x = \frac{y^2}{2s}$. In the Cartesian plane, both of these curves are parabolas (one opening upward and the other opening rightward). These intersect, as shown in Note 4.5.A, when $x = \sqrt[3]{2}s$ (and hence $x^3 = 2s^3$, as desired). The second solution is considered in the history component of Introduction to Modern Geometry (MATH 4157/5157) on Chapter 3. Conic Sections (see Note 3.A). We now give a slightly modified presentation of the same argument. The two mean proportionals problem of Note 4.5.A, $\frac{s}{x} = \frac{x}{y} = \frac{y}{2s}$, implies $y = \frac{x^2}{s}$ and $y = \frac{2s^2}{x}$. In the Cartesian plane, $y = \frac{x^2}{s}$ is a parabola (opening upward) and $y = \frac{2s^2}{x}$ is a hyperbola with the coordinate axes as asymptotes (it is a rotation through $\pi/4$ of a hyperbola with a horizontal axis). These also intersect, as shown in Note 4.5.A, when $x = \sqrt[3]{2}s$ (and hence $x^3 = 2s^3$, as desired). Of course, Menaechmus did not use the terms "parabola" or "hyperbola," since this is terminology introduced by Apollonius of Perga (circa 262 BCE–circa 190 BCE). Now Meneachmus cannot describe conic sections in terms of Cartesian products (not can Apollonius), since such things do not appear until René Descartes (March 31, 1596–February 11, 1650) introduces them in his 1637 *La Géométrie*. Instead, he introduces these curves using the definitions giving the curves as the locus of all points whose distance from a fixed point (called the *focus*) is a constant multiple (called the *eccentricity*) of the distance to a fixed line (called the *directrix*). When

$0 < e < 1$ the conic is an ellipse, when $e = 1$ is is a parabola, and when $e > 1$ is is a hyperbola. Heath gives a description of Menaechmus' solution (see his pages 254 and 255), where the semi latus rectum plays a prominent role.

Note 4.5.C. To illustrate the use of idealized mechanical devices in the solution of a construction problem, we consider the solution credited to Plato. We follow Eves' presentation on this, who comments (see page 112): "...it is known that Plato objected to such methods [i.e., mechanical methods], it is felt that the ascription to Plato is erroneous."

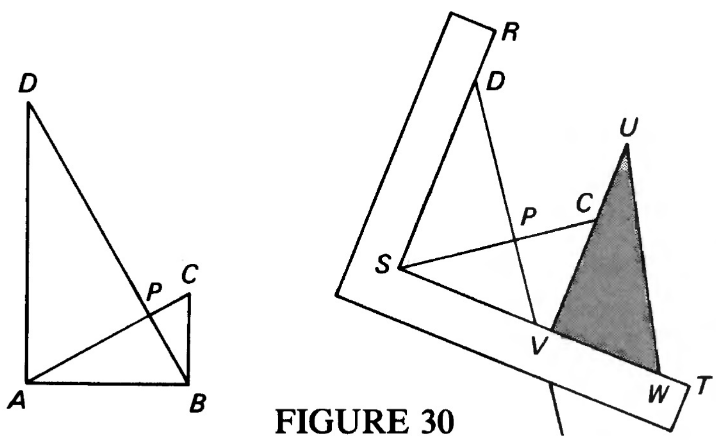


FIGURE 30

Consider the two triangles of Figure 30 left. We take triangles CBA and DAB as right triangles with the right angles as B and A, respectively. The triangles share leg $\overline{AB}$. Let the hypotenuses $\overline{AC}$ and $\overline{BD}$ intersect perpendicularly at point P, as shown. Now triangles CPB and BPA are similar because: the angle at B in CPB is the complement of the angle at B in BPA, and the angle at C is the complement of the angle at B in CPB, so the angle at C equals the angle at B in BPA. Therefore corresponding angles in CPB and BPA are equal, and these triangles are similar. Also triangles APD and CPB are similar: the angle at C equals the angle at A in APD since they are alternating interior angles resulting from a transversal which cuts parallel lines, and similarly the angle at D equals the angle at B in CPB. Therefore corresponding angles in CPB and APD are equal, and these triangles are similar (and also similar to BPA). So corresponding edges of these three triangles are in equal proportions and we have $PC : PB = PB : PA = PA : PD$. So, in the terminology of Menaechmus, PB and PA are the two mean proportional between PC and PD. If the figure can be constructed whee $PD = 2(PC)$, then we have a doubling of the cube (because we will then have constructed $\sqrt[3]{2}$, as argued in Notes 4.5.A and 4.5.B).

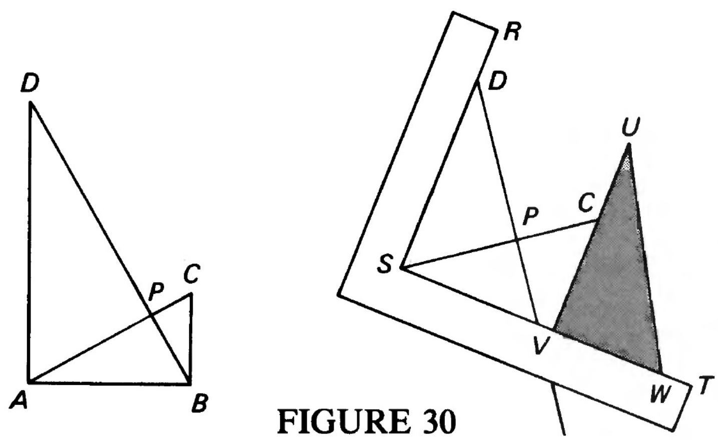


FIGURE 30

Now we construct the "mechanism" as given in Figure 30 right. First, draw two perpendicular line segments $\overline{PC}$ and $\overline{PD}$ intersecting at P and with $PD = 2(PC)$. Second, place a "carpenter's square" (as Eves says, page 112; we just need two rays that meet at a right angle) with inner edge RST on the figure so that $\overrightarrow{SR}$ passes through D and the vertex S of the right angle lies on $\overline{CP}$ extended. On $\overrightarrow{ST}$, place a right triangle (or another "carpenter's square") UVW with leg $\overline{VW}$ on $\overrightarrow{ST}$. Slide triangle UVW along $\overrightarrow{ST}$ until leg $\overline{VU}$ passes through C, as in Figure 30 right. Finally, "manipulate" the device until point V falls on $\overline{DP}$ extended. The manipulation is done, in the case given in Figure 30 right, by sliding the carpenter's square further along $\overline{CP}$ extended (i.e., move point S further from point P while keeping S on $\overline{CP}$ extended) while keeping fixed point D on $\overrightarrow{SR}$. This results in a sort-of clockwise rotation of the carpenter's square that will move V closer to S on $\overrightarrow{ST}$ with it eventually lying on $\overline{DP}$ extended. This results in a configuration as given in Figure 30 left, with A replaced by S, B replaced by V, and $PD = 2(PC)$, as required.

Note. Some additional constructions are given in the Chapter 4 problems. Problem 4.2 addresses the solutions of Archytas and Menaechmus, Problem 4.3 covers solutions of Apollonius and Eratosthenes, and Problem 4.4 gives Diocies solution using the cissoid.

Note. We now give a modern resolution of the duplication of the cube. Effectively, this requires establishing the existence of $\sqrt[3]{2}$. The approach taken when using only Euclidean tools is to give a straight edge and compass construction of $\sqrt[3]{2}$ which, as we observed above, does not exist. So how do we really know that this number exists? The same question holds for any root of a positive real number. We might observe that there is a number a which, when cubed, is less than 2: $a^3 < 2$ (we could take $a = 1$ for example, or $a = 5/4$; $1^3 = 1$ and $(5/4)^3 = 125/64 = 1.953125$). There is also a number b which, when cubed, is greater than 2: $b^3 > 2$ (we could take $b = 2$ or $b = 63/50$; $2^3 = 8$ and $(63/50)^3 = 250047/125000 = 2.000273$). So there must be some number c between a and b, $a < c < b$, such that $c^3 = 2$.

But this requires some kind of concept of a continuum (a concept not rigorously developed until the 19th century). We see in Note 4.5.C that the sliding around of the second carpenter's square is equivalent to an assumption of a continuum. The continuum idea is addressed in Analysis 1 (MATH 4217/5217) with the Axiom of Completeness for the real numbers $\mathbb{R}$. It states that every set of real numbers with an upper bound has a least upper bound. For details, see my online notes for Analysis 1 on Section 1.3. The Completeness Axiom. The cube root of 2 is then defined as the least upper bound (denoted "lub") of a set with an upper bound. The formal definition is: $\sqrt[3]{2} = \text{lub}\{x \mid x^3 < 2\}$. An upper bound is given by $b = 2$, for example (or $b = 63/50$) and the existence of the least upper bound is then given by the Axiom of Completeness. (It is the Axiom of Completeness that *makes* the real numbers a continuum.) More generally, for $c > 0$ a positive real number and n a positive integer, we define $\sqrt[n]{c} = \text{lub}\{x \mid x^n < y\}$ (this is Theorem 1-8 in the Analysis 1 notes on Section 1.2. The Real Numbers, Ordered Fields and a proof is to be given in Exercise 1.3.9 in that class that $(\sqrt[n]{c})^n = c$, as desired). The least upper bound of a set is unique, so this analytic approach gives the existence of a unique positive nth root of a positive real number. With this defined, for $c > 0$ and for positive rational exponent p/q we now have $c^{p/q} = (\sqrt[q]{c})^p$. Least upper bounds are needed in defining the value of a positive real number to an *irrational* power (see the Section 1.3 Analysis 1 notes for details).

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