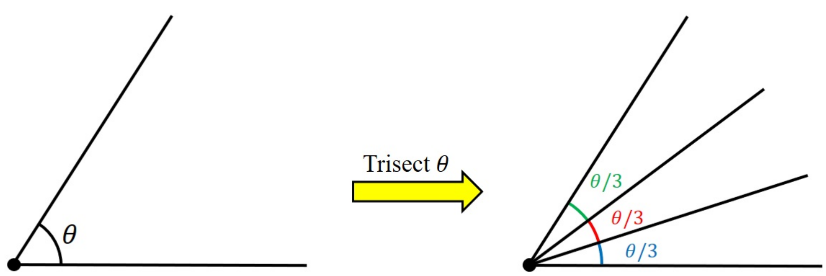
4.6. Trisection of an Angle

Note. In this section we discuss the impossibility of a straight edge and compass construction that allows us to break *any* angle into three equal parts. Of course, some angles can be trisected with Euclidean tools; a 90° angle can be trisected because a 60° angle can be constructed (Euclid's Proposition I.1 is the construction of an equiliateral triangle) and an angle can be bisected with Euclidean tools. But there are angles that cannot be trisected with straight edge and compass. A 60° angle cannot be trisected because a 20° angle is not constructible. We will give some history of the problem and consider some additional tools that allow a solution to the problem.



Note. Eves states (see pages 112 and 113): "Of the three famous problems of Greek antiquity, the trisection of an angle is pre-eminently the most popular among the mathematically uninitiated in America today. . . . The problem is certainly the simplest one of the three famous problems to comprehend, and since the bisection of an angle is so very easy, it is natural to wonder why trisection is not equally easy." I, "your humble instructor," was taken in by this problem when, in 10th grade, I took high school geometry (during the 1978-79 academic year; for sentimental reasons, I have a few online notes based on the book we used). Having been told that an angle could not be trisected with a straight edge and compass, I skeptically set off to work on it! My reasoning went (as I recall, now 45 years on): Bisection of an angle, as Eves mentions, is straightforward (it is Euclid's Proposition I.9 in the *Elements*; it can be accomplished by bisecting a cord associated with an arc along the angle) and a line segment can easily be trisected (or, for that matter, cut into any number of equal-length parts), so some adaptation of the trisecting a line segment should be applicable to the trisecting of an angle. I did not spend endlessly hours on it, but I did eat up several evenings on it. (Spoiler Alert: I did not find a successful trisection construction!)

Note 4.6.A. As with doubling the cube, angle trisection cannot (in general) be done with only a straight edge and compass. Again, recall that the constructible numbers are precisely the real numbers that can be obtain from $\mathbb{Q}$ by taking square roots of positive numbers a finite number of times and applying a finite number of field operations (see Note 3.7.F of Section 3.7. Geometric Solution of Quadratic Equations). This is Theorem 32.6 in my online notes for Introduction to Modern Algebra 2 (MATH 4137/5137) on Section VI.32. Geometric Constructions. As shown in those notes (see the proof of Theorem 32.11, "Trisecting the Angle is Impossible"), a 60° cannot be trisected because the construction of a 20° is equivalent to the construction of the number $\cos(20^\circ)$, and this number satisfies the polynomial equation $8x^3 - 6x - 1 = 0$ where $x = \cos(20^\circ)$. However, this polynomial is irreducible over the rationals, which implies that the solution involves a nonconstructible number involving cube roots (the history of this algebraic argument by Pierre Wantzel in 1837 is given in Section 14.2. Impossibility of Solving the Three Famous Problems with Euclidean Tools). In fact, the online computer algebra system Wolfram Alpha gives the exact value as

$$\cos(20^\circ) = -\frac{1 - i\sqrt{3}}{\sqrt[3]{\frac{1}{2}(1 + i\sqrt{31})}} - \frac{1}{2}(1 + i\sqrt{3})\sqrt[3]{\frac{1}{2}(1 + i\sqrt{31})}.$$

This value is real, even though there are imaginary units present in this version; we'll see more of this when we study solutions to third and fourth degree polynomial equations in Section 8.8. Cubic and Quartic Equations.

Note. With the impossibility of a straight edge and compass trisection construction established in the 1800s as a result of the introduction of modern algebra and field theory, there is now no need to work on the problem! None-the-less, there are still occasional attempts to find such a construction (and even claims of success). Underwood Dudley (January 6, 1937–) refers to those who make such attempts in the face of the impossibility of success as "cranks." He has a book devoted to those who continue to make such attempts. In his *A Budget of Trisections* (Springer, 1987), he describes his attempts to address this: "What follows, then, is something which has never been done before: it is an effort to do something which may be as impossible as trisecting the angle, namely to put an end to trisections and trisectors." See his page xv. The cover of his book has an amusing Don Quixote-motivated take on the "trisectors":

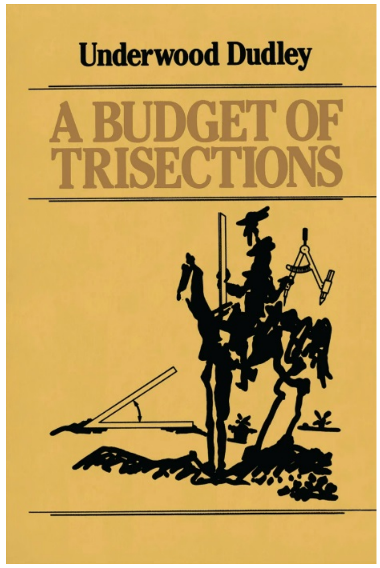


Image from the Spinger.com website (accessed 3/18/2023)

Note 4.6.B. Consider the acute angle $\angle ABC$ taken as the angle between a diagonal $\overline{BA}$ and a side $\overline{BC}$ of rectangle $BCAD$ (see Figure 31). Consider a line segment starting at point B which intersects $\overline{AC}$ at, say E, intersects segment $\overline{DA}$ extended at a point F, and such that $EF = 2(BA)$. Think of segment $\overline{BF}$ as rotating about the point B as point F moves along $\overline{DA}$ extended; when point F is close to point A then length BF is close to the length BA (and slightly bigger when F is to the right of A). As point F moves farther to the right, the length of BF gets larger and larger without bound. So there exists some point F where the length of $\overline{BF}$ is twice the length of $\overline{BA}$; this is a *continuity* argument and not a construction argument based on the use of Euclidean tools. The Greeks called this a *verging problem* because the line segment $\overline{FE}$ "verges" toward point B.

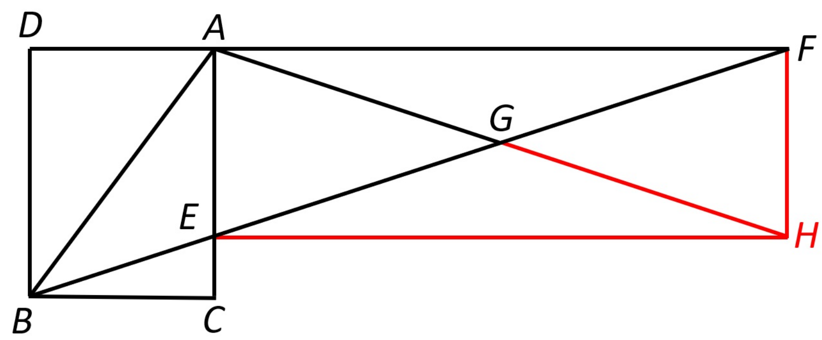
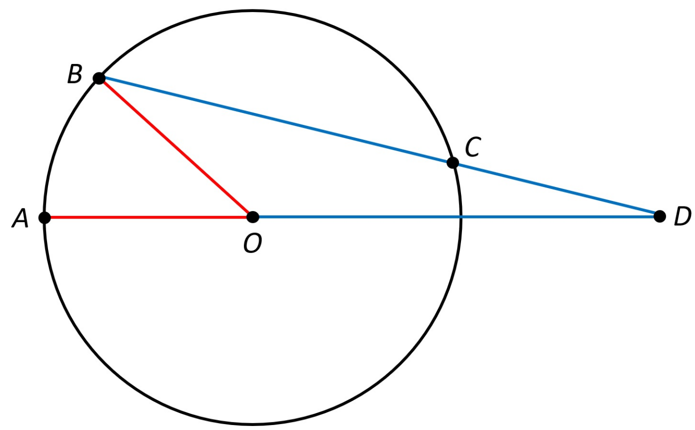


Figure 31 (Modified)

Next, let G be the midpoint of $\overline{EF}$ and insert segment $\overline{AG}$. By the choice of points E, F, and G we have the lengths $EG = GF = BA$. By inserting point H, $\overline{FH}$, and $\overline{EH}$ to create rectangle $AFHE$ and extending $\overline{AG}$ to give diagonal $\overline{AH}$ of the rectangle (in red in Figure 31 above), we see $AG = GF$. Therefore $EG = GF = GA = BA$. So triangle ABG is isosceles and the measure of its angles satisfy $\measuredangle ABG = \measuredangle AGB$. Now angles $\angle AGB$ and $\angle AGF$ are supplements, and $\measuredangle GAF + \measuredangle GFA + \measuredangle AGF = 180°$, so we have $\measuredangle ABG = \measuredangle AGB = \measuredangle GAF + \measuredangle GFA$. Since triangle AFG is isosceles, then $\measuredangle GAF = \measuredangle GFA$, and hence $\measuredangle ABG = \measuredangle GAF + \measuredangle GFA = 2\measuredangle GFA$. By construction, we have $\measuredangle ABC = \measuredangle ABG + \measuredangle GBC$, or $\measuredangle ABC = 2\measuredangle GFA + \measuredangle GBC$ (since $\measuredangle ABG = 2\measuredangle GFA$). Now $\measuredangle GFA = \measuredangle GBC$ since these are alternate interior angles of parallel lines cut by a transversal), so the last equation implies or $\measuredangle ABC = 3\measuredangle GBC$. That is, line segment $\overline{BG}$ trisects $\angle ABC$. Thomas Heath in his *A History of Greek Mathematics, Volume I. From Thales to Euclid* (Clarendon Press, Oxford, 1921) shows that the construction above can be produced using a hyperbola (a construction which he credits to Pappus) and that it is equivalent to solving a cubic equation (see his pages 236–238).

Note 4.6.C. In Problem 4.6(b), the following trisection of an angle is described. Let $\angle AOB$ be any central angle in a given circle. Through point B, draw a line $\overline{BCD}$, cutting the circle again in C, and cutting $\overline{AO}$ extended at point D, such that $CD = OA$ (the radius of the circle). Then $\measuredangle ADB = \frac{1}{3}\measuredangle AOB$. The configuration described here is:



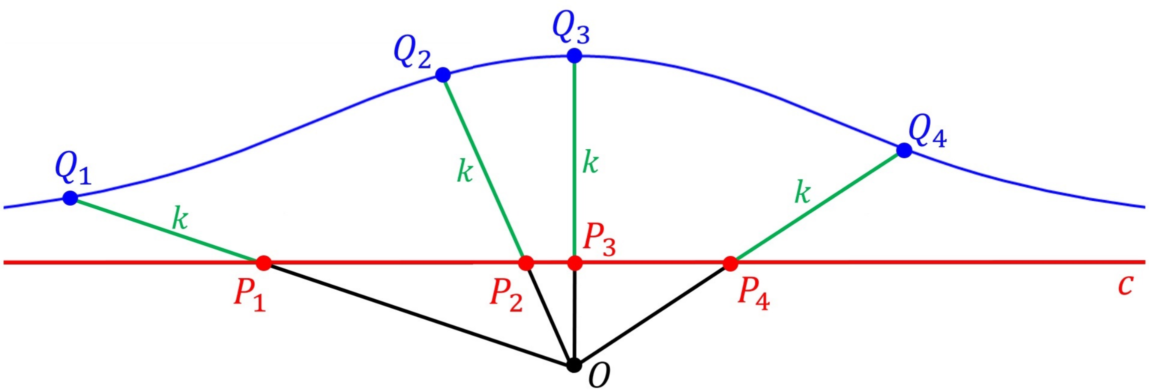
Just as in Note 4.6.B, this construction requires the placement of a line segment of a given length on a line through a given point (think of the straight edge as rotating around the given point until the desired length is attained). The ability to do this requires a ruler to measure the desired distance, and the act of "rotating" the straight edge is not permitted in the use of Euclidean tools. So this component of the constructions do not adhere to the straight edge and compass constructions using Euclidean tools. This new manipulation is called the *insertion principle*.

Note. Nicomedes (circa 280 BCE–circa 210 BCE) is known only from references to his work by others. It is known that he criticized work of Erathosthenes (276 BCE–194 BCE) on the duplication of the cube, and that Apollonius (262 BCE–190 BCE) named a curve the "sister of the conchoid [of Nicomedes]." These references are the sources of the estimated dates of Nicomedes. Second-hand references to his work reveal that he wrote *On Conchoid Lines*.

Image from the MacTutor History of Mathematics Archive biography of Nicomedes (accessed 4/25/2023)

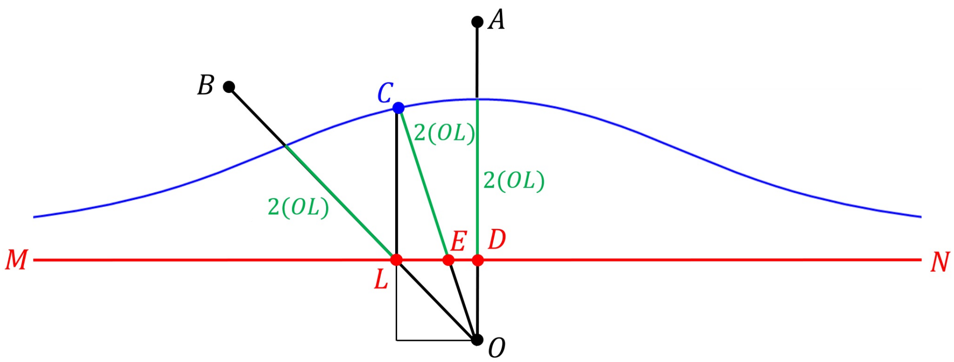
The *conchoid of Nicomedes* is described in a major work of Pappus of Alexandria (circa 290 CE–circa 350 CE). Pappus (who we discuss in Section 6.9. Pappus) wrote *Synagoge* or the *Mathematical Collection* as a work covering practically the whole field of classical Greek geometry. Book 4 of this work is still in print in English as *Pappus of Alexandria: Book 4 of the Collection: Edited With Translation and Commentary by Heike Sefrin-Weis* (Springer, Sources and Studies in the History of Mathematics and Physical Sciences, 2010). The conchoid of Nicomedes is described in Book 4, Chapters 26 and 27 (in Chapters 28 and 29 Pappus shows how the conchoid can be used to double the cube); see page 379 of Thomas Heath's *A History of Greek Mathematics, Volume II: From Aristarchus to Diophantus*, Clarendon Press, Oxford, 1921 (reprinted by Dover Publications, 1981). We now turn to Eves' description of the conchoid.

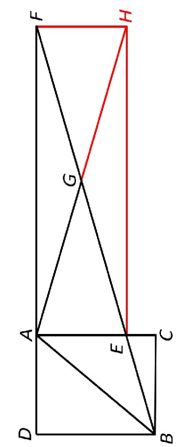
Note. Let c be a line and O a point not on c. For each point P on line c, extend line segment $\overline{OP}$ a fixed length k to create line segment $\overline{OQ}$. Then the locus of points Q that results is a *conchoid* of c for the pole O and the constant k (Eves says this is "one branch" of the conchoid; a second branch results when the fixed length k to point Q goes in the opposite direction from point P back towards point O). See the figure below.



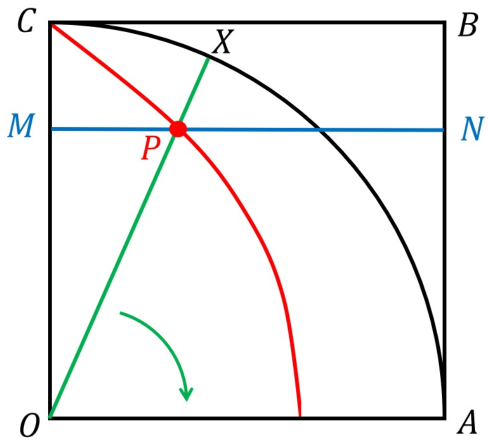
A mechanical device (based on an illustration in Eutocius of Ascalon's [circa 480–circa 540] commentaries on the works of Archimedes) that generates a conchoid is illustrated on the Wikipedia page on the Conchoid (accessed 4/26/2023). Recall that the relationships between rectangular coordinates (x, y) and polar coordinates (r, θ) are $x = r\cos\theta$ and $y = r\sin\theta$. So with point O as the origin and the distance from O to line c as ℓ, line c has equation $y = \ell$ in rectangular coordinates and equation $r\sin\theta = \ell$, where $\theta \in (0, \pi)$, in polar coordinates. That is, line c is represented as $r = \ell\csc\theta$ in polar coordinates. Since points on the conchoid are an additional distance k from line c (where distance is measured from O), then we simply increase r by k to get the polar coordinate form of the conchoid as $r = k + \ell\csc\theta$ where $\theta \in (0, \pi)$.

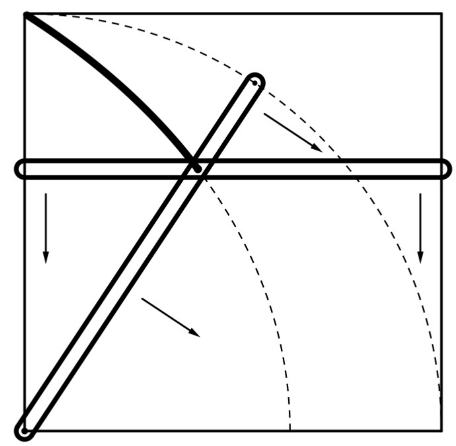
Note 4.6.D. Let $\angle AOB$ be any given acute angle. Draw line $\overleftrightarrow{MN}$ perpendicular to $\overline{OA}$ and intersecting $\overline{OA}$ and $\overline{OB}$ at points D and L, respectively. Introduce the conchoid of $\overleftrightarrow{MN}$ for pole O and constant $k = 2(OL)$. At point L draw the parallel to $\overline{OA}$ and let C be the point at which the parallel intersects the conchoid. See the figure below (left, which is a modified version of Eves' Figure 32). In Figure 31 of Note 4.6.B (reproduced below, right) we have $EF = 2(BA)$ and $\measuredangle FBC = \frac{1}{3}\measuredangle ABC$. Correspondingly we have from the conchoid (below, left) that $EC = 2(OL)$ and hence, as in Figure 31, $\measuredangle COA = \frac{1}{3}\measuredangle AOB$.





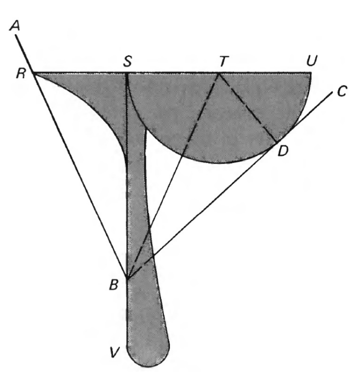
Note 4.6.E. Hippias of Elis (circa 460 BCE–circa 400 BCE) is credited by Proclus (circa 411–April 17, 485), in his *A Commentary on the First Book of Euclid's Elements* (described in part in Supplement. Proclus's Commentary on Eudemus' History of Geometry) with introducing the *quadratrix*. The quadratrix is defined in Eves' Problem 4.10 as follows. Let the radius $\overline{OX}$ of a circle rotate uniformly about the center O from $\overline{OC}$ to $\overline{OA}$, where $\overline{OA}$ is at a right angle to $\overline{OC}$. At the same time, let a line segment $\overline{MN}$ parallel to $\overline{OA}$ move uniformly parallel to itself from $\overline{CB}$ to $\overline{OA}$. The locus of the intersection P of $\overline{OX}$ and $\overline{MN}$ is the quadratrix.

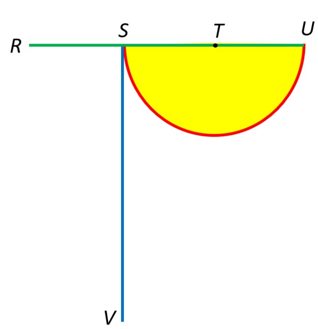


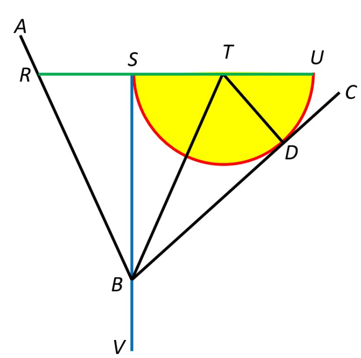


(The image on the right gives a mechanical device that generates a drawing of a quadratrix. It is from the Wikipedia page on the Quadratrix of Hippias; accessed 4/29/2023.) Since $\overline{OA}$ rotate uniformly and $\overline{MN}$ moves uniformly (at the same time), then the distance OM is proportional to $\measuredangle AOX$. This makes it straightforward to trisect an angle (or "multisect," or even to create any multiple m of a given angle where m is constructible and $0 < m < 1$); this is to be done in Problem 4.10(b). Pappus of Alexandria (circa 290 CE–circa 350 CE) gives a detailed description of the quadratrix in Book 4 of his *Mathematical Collection* (mentioned above in connection with the conchoid of Nicomedes); see also pages 226–230 of Thomas Heath's *A History of Greek Mathematics, Volume I: From Thales to Euclid*, Clarendon Press, Oxford, 1921. Though the quadratrix is commonly attributed to Hippias, there is not a consensus that Hippias used it to trisect and angle (see Heath's page 226 for references). The quadratrix can also be used to square a circle ("quadrature of the circle"), as is to be shown in Problem 4.10(c). It is also unclear whether or not this result can be attributed to Hippias.

Note. We have seen the use of "mechanical devices" to duplicate the cube (in Note 4.5.C) and to trisect an angle with a mechanically constructed conchoid (in Note 4.5.E; the conchoid also allows for the squaring of the circle as shown in Problem 4.10(c)). A number of such mechanical devices that allow for constructive solutions of the "Three Famous Problems." A survey of such devices is given by Robert Yates in "The Trisection Problem: Chapter III Mechanical Trisectors," *National Mathematics Magazine*, **15**(6), 278–293 (1941). This source includes the use of a cone to trisect an angle, as described in Problem 4.8(c) (see Yates' Figure 31). Eves mentions the "so-called *tomahawk*," which is given in Yates' Section 7 and Figure 21. It is not known who came up with the idea of the tomahawk, but the oldest known reference is in the third edition of Bergery's *Geométrie appliquée a l'industrie*, Metz (1835). Eves gives an image of the tomahawk in his Figure 33 (below, left). His image includes some unnecessary parts (likely, because he bases his Figure 33 on Yates' Figure 21), which makes the structure look more like a tomahawk (a tomahawk is a small axe). Below center is a more schematic image of the tomahawk which only includes the necessary parts.







The construction of a tomahawk starts with line segment $\overline{RU}$, which is trisected at points S and T. Introduce a semicircle centered at T with diameter $\overline{SU}$, and $\overline{SV}$ perpendicular to $\overline{RU}$, as shown above. The tomahawk is used to trisect angle

$\angle ABC$ by placing point R of the tomahawk on $\overline{BA}$, arranging $\overline{SV}$ such that it contains vertex B, and adjusting the tomahawk until the semicircle is tangent to $\overline{BC}$ at point D, as shown above (left and right). We now have three right triangles, $\triangle RSB$, $\triangle TSB$, and $\triangle TDB$. Now $\triangle RSB$ and $\triangle TSB$ are congruent (by SAS) since they share edge $\overline{SB}$ and $RS = ST$ (because of the trisection of $\overline{RU}$). Next, $\triangle TSB$ and $\triangle TDB$ are congruent (by SSS) since they are right triangles which share their hypotenuses and $ST = TD$ (because both $\overline{ST}$ and $\overline{TD}$ are radii of the semicircle; the length of the third sides follow from the Pythagorean Theorem). Therefore, $\measuredangle RBS = \measuredangle SBT = \measuredangle TBD$. Since $\measuredangle RBS + \measuredangle SBT + \measuredangle TBD = \measuredangle ABC$, then $\angle ABC$ is trisected by each of $\angle RBS$, $\angle SBT$, and $\angle TBD$. Eves claims (see his page 115) that with two tomahawks, we can quintisect an angle.

Note. We'll see in the next section, Section 4.7. Quadrature of the Circle, that the Archimedean spiral can also be used to trisect an angle (or, as with the quadratrix of Hippias, to multisect an angle).

Note 4.6.F. Since we cannot precisely trisect and angle with a compass and straight edge (in a finite number of steps), then attention turns to approximations to a trisection. We know that an angle can be bisected with a compass and straight edge (Euclid's Proposition I.9), so we can simply use a sequence of bisections to approximate the trisection (in fact, we can precisely trisect an angle with a compass and straight edge *in the limit*; that is, by performing an *infinite* number of bisections). We can write 1/3 in binary and use the sequence of 0's and 1's to determine the angles to be bisected. This idea is given in Nikolais Fialkowski's *Teilung des Winkels und des Kreises* ["Division of the Angle and the Circle"] (1860); see page 11. It is mentioned in Robert Yates' "The Trisection Problem: Chapter IV Approximations," *National Mathematics Magazine*, **16**(1), 20–28 (1941), where the following figure appears:

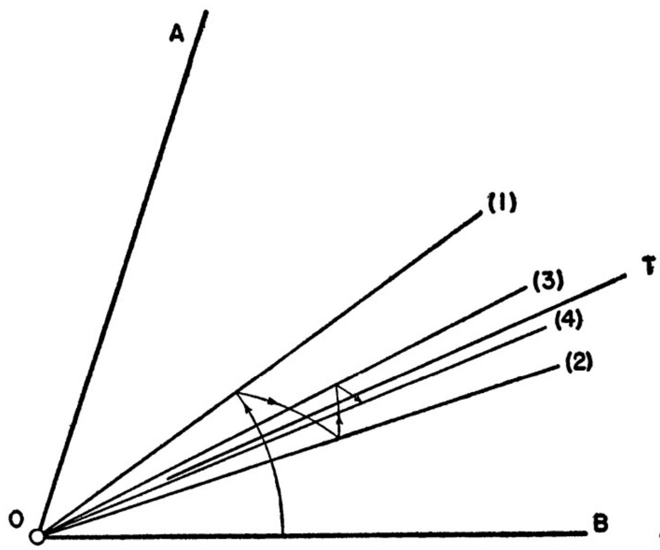


FIG. 32

We have base 2 that $1/3 = (0.01010101\cdots)_2$. Interpret the figure above as: bisect $\angle AOB$ with (1), bisect $\angle(1)OB$ with (2) (because the first digit of 1/3 base 2 is 0), bisect $\angle(1)O(2)$ with (3) (because the second digit of 1/3 base 2 is 1), and bisect $\angle(3)O(2)$ with (4) (because the third digit of 1/3 base 2 is 0). Since the digits of 1/3 base 2 alternate between 0 and 1, we continue to bisect angles alternately "lower angle" and "upper angle" resulting, in the limit, in the trisection of $\angle AOB$. In this way, we can get any desired level of precision (other than equality) in the estimation of $\frac{1}{3}\measuredangle AOB$ in a finite number of steps. This is explored in Problem 4.9(a).

Note. Another example of of an approximation technique, but only requiring a few steps, is due to the painter Albrecht Dürer, appearing in his *Unterweysung der meesung mit dem zierkel und richtscheyt* [Instructions for Measuring with Compass and Straight Edge], Nürnberg (1525). The technique is given in Figure 34 of Eves:

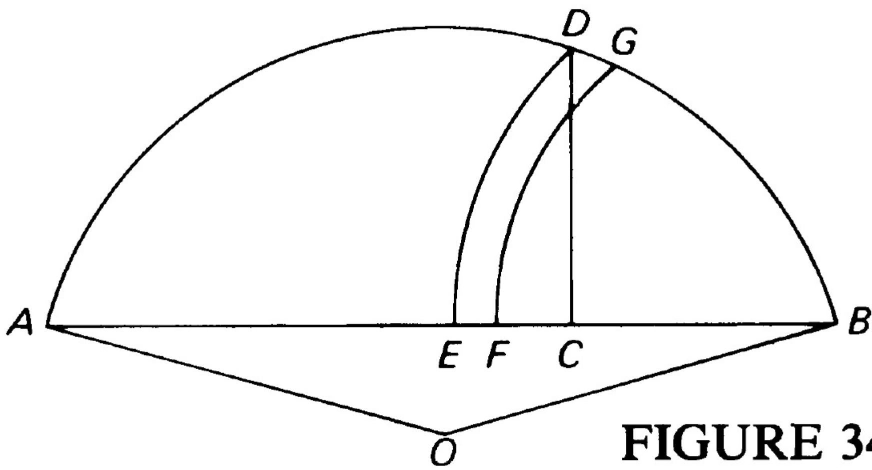


FIGURE 34

Let $\angle AOB$ be a given central angle in a circle. Let point C be that trisection point of chord $\overline{AB}$ that is nearer point B. At point C construct the perpendicular to $\overline{AB}$ and let D be the point at which is intersects arc $\widehat{\mathrm{AB}}$ in the given circle. With B as the center and BD as radius, insert an arc to cut $\overline{AB}$ at point E. Let F be the trisection point of $\overline{EC}$ that is nearer to E. Again with B as the center and BF as the radius, insert a arc to cut $\overline{AB}$ at point G. Then $\overline{OG}$ is an approximate trisecting line of $\angle AOB$. This approximation technique is explained in Section 3 of Robert Yates' "The Trisection Problem: Chapter IV Approximations," *National Mathematics Magazine*, **16**(1), 20–28 (1941). Yates show that with $\theta = \frac{1}{2}\measuredangle AOB$, the estimate of $\frac{1}{3}\measuredangle AOB$ is

$$2\sin^{-1}\left(\frac{1}{9}\sin\theta + \sqrt{\frac{2}{27}}\sqrt{2+\cos^2\theta - \cos\theta\sqrt{8+\cos^2\theta}}\right).$$

Yates then gives the following table of values of $\measuredangle AOB$ and the error of approximating $\frac{1}{3}\measuredangle AOB$ with θ:

$\measuredangle AOB$	Error	$\measuredangle AOB$	Error
60°	$1''$	140°	$5'37''$
90°	$18''$	150°	$9'4''$
120°	$11'56''$	180°	$31'38''$

Note. We now give a modern resolution of the trisection of an angle, and of the existence of any size angle in general. This is covered in Introduction to Modern Geometry (MATH 4157/5157). In this discussion we follow my online notes for Introduction to Modern Geometry on Section 2.6. Angles and Angle Measurement. First, an *angle* is defined as the union of two rays which have a common endpoint and do not lie on the same straight line (notice that this definition does not allow for "zero angles" nor "straight angles"). Postulates are then given that allow us to associate a real number with an angle. First, a positive number R is assigned to a straight angle and this allows us to assign a number to every angle (as defined here):

> **Postulate 13.** If R is any positive number, there exists a correspondence which associates with each angle in space a unique positive number between 0 and R.

The number R of Postulate 13 is the *scale factor* and the number assigned to a given angle is the *measure of the angle relative to the scale factor*. We denote the measure of angle $\angle ABC$ and $\measuredangle ABC$. In the event that the scale factor is $R = 180$ then we are measuring angles in degrees, and if $R = \pi$ then we are measuring angles in radians. In Postulate 14 of the online notes it is given that, for different scale factors R and S, we can convert the measure of an angle relative to S to the measure relative to R by multiplying by R/S. In the next postulate, a one-to-one correspondence between the numbers in $[0, R]$ and the angles between a zero angle and a straight angle is hypothesized.

Postulate 15. The Protractor Postulate. If H is any halfplane, $\overrightarrow{VA}$ any ray lying in the edge of H, and R any positive number, there is a one-to-one correspondence between the set of all numbers x for which $0 \leq x \leq R$ and the set of rays, $\overrightarrow{VX}$ which lie in the union of H and its edge, such that:

(1) $\overrightarrow{VA}$ corresponds to the number 0,

(2) the ray opposite to $\overrightarrow{VA}$ corresponds to the number R, and

(3) if X and Y are not collinear with V and if x and y are the numbers which correspond to $\overrightarrow{VX}$ and $\overrightarrow{VY}$, respectively, then $\measuredangle XVY = |x - y|$.

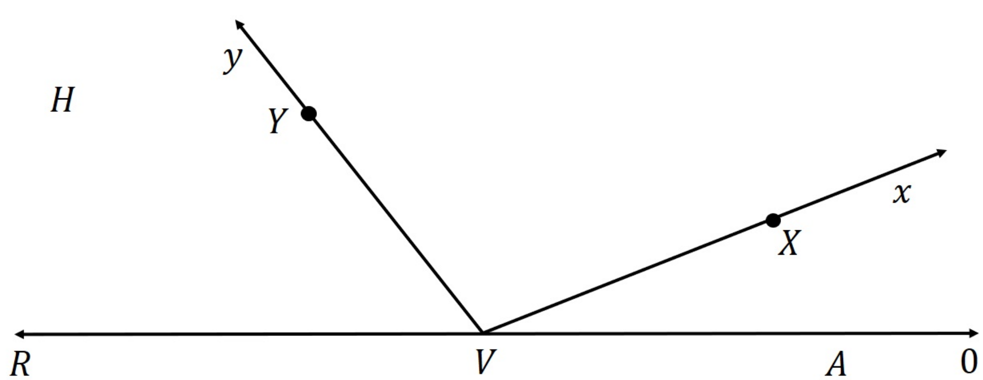


Figure 2.13 from C. R. Wylie's *Foundations of Geometry*, McGraw-Hill (1964)

Notice that the Protractor Postulate allows to trisect any angle with measure $m \in [0, R]$ (and hence to trisect any measure angle) by finding an angle with measure

$m/3 \in [0, R/3]$. Of course this hypothesis gives us the more general property that we can create an obtuse angle of *any* measure in $[0, R]$. As in Section 4.5. Duplication of the Cube, we have introduced a continuum here. By postulating the one-to-one correspondence between angles and the interval of real numbers $[0, R]$, we have imposed the existence of a continuum of angles. When restricted to straight edge and compass constructions, one can only construct points in $\mathbb{R}^n$ which have constructible coordinates (hence can only construct certain angles and cannot construct a continuum of angles).

Revised: 5/11/2023