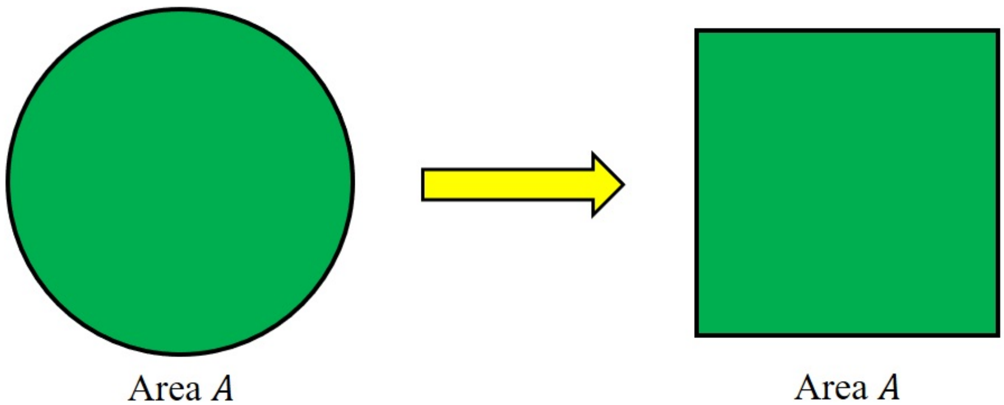
4.7. Quadrature of the Circle

Note. In this section, we discuss the third of the "Three Famous Problems" from Section 4.3. Namely, we consider constructing with a compass and straight edge a square having an area equal to that of a given circle (where the given circle has a constructible radius).



Note 4.7.A. As with doubling the cube and trisecting the angle, squaring the circle (or "quadrature of the circle") cannot be done with only a straight edge and compass. Again, recall that the constructible numbers are precisely the real numbers that can be obtain from $\mathbb{Q}$ by taking square roots of positive numbers a finite number of times and applying a finite number of field operations (see Note 3.7.F of Section 3.7. Geometric Solution of Quadratic Equations). This is Theorem 32.6 in my online notes for Introduction to Modern Algebra 2 (MATH 4137/5137) on Section VI.32. Geometric Constructions. We have seen that doubling the cube is equivalent to constructing $\sqrt[3]{2}$, but $\sqrt[3]{2}$ does not satisfy Theorem 32.6 (the numbers satisfying this are called "constructible"; these ideas were given by Pierre Wantzel

in 1837, as mentioned in connection with the other two Famous Problems). Trisecting a 60° angle is equivalent to constructing a solution to the polynomial equation $8x^3 - 6x - 1 = 0$ ($x = \cos 20^\circ$ is the one positive solution; there are also two negative solutions). Solutions to this equation to not satisfy Theorem 32.6 either. So the inability to do either of these problems can be explained using roots of third degree polynomials. More details on this are given in Section 14.2. Impossibility of Solving the Three Famous Problems with Euclidean Tools; ultimately the explanation is based on the fact each constructible number is contained in a power of two degree extension field of the rational numbers. The "power of two" conditions is violated by the third degree polynomials associated with both doubling the cube and trisecting the (60 degree) angle. The situation is trickier for the quadrature of the circle. This problem is equivalent to constructing π with a compass and straight edge. Johann Lambert (August 26/28, 1728–September 25, 1777) proved that π is irrational in "Mémoire sur quelques proprétés remarquables des quantités transcendantes, circulaires et logarithmiques, *Mémoires de l'Académie Royale des Sciences de Berlin*, 265–322 (1768). However, there are irrational numbers that are roots of second degree polynomials (such as $x = \sqrt{2}$, a root of $x^2 - 2 = 0$). So this is not sufficient to show that π is not constructible. A real number is *algebraic* if it is a zero of a polynomial with integer coefficients. For example, $x = \sqrt[3]{2}$ is algebraic (but not constructible) since it is a zero of the polynomial $p(x) = x^3 - 2$. All constructible numbers are algebraic (a number is constructible if it is a zero of a *certain kind* of polynomial), but not all algebraic numbers are constructible (such as $\sqrt[3]{2}$). If one could show that π is not algebraic, then it would follow that π is not constructible and that it is impossible to square the circle with a compass and straight edge. A real number that is not algebraic is called *transcendental.* Ferdinand von Lindemann (April 12, 1852–March 6, 1939) proved that π is transcendental in Über die Zahl π [On the Number π]," *Mathematische Annalen*, 20, 213–225 (1882). With this, the impossibility of a compass and straight edge solution any of the Three Famous Problems was established. The Three Famous Problems predate Euclid (circa 325 BCE–circa 265 BCE), so these math problems remained unsolved for over 2200 years!

Note. The first Greek known to be connected to the squaring of the circle is Anaxagoras of Clazomenae (499 BCE–428 BCE). According to Plutarch (45 CE–119 CE) in his *On Exile*, Anaxagoras wrote on the squaring of the circle while in prison (he was imprisoned for claiming that the Sun was not a god), though none of this writing survives. This information is from the MacTutor History of Mathematics Archive biography of Anaxagoras and the MacTutor History of Mathematics Archive page on Squaring the Circle (accessed 5/10/2023).

Note. Hippocrates of Chios (circa 470 BCE–circa 410 BCE; not to be confused with Hippocrates of Cos, the physician), wrote the first geometry textbook. He considered the areas of "lunes" based on circles with diameters associate with a right triangle. This problem partially concerns expressing areas of circles in terms of the area of a right triangle, so it is related to the problem of squaring the circle. More details on the lunes of Hippocrates are given in the Introduction to Modern Geometry (MATH 4157/5157) class in Section 1.8. Three Famous Problems of Greek Geometry

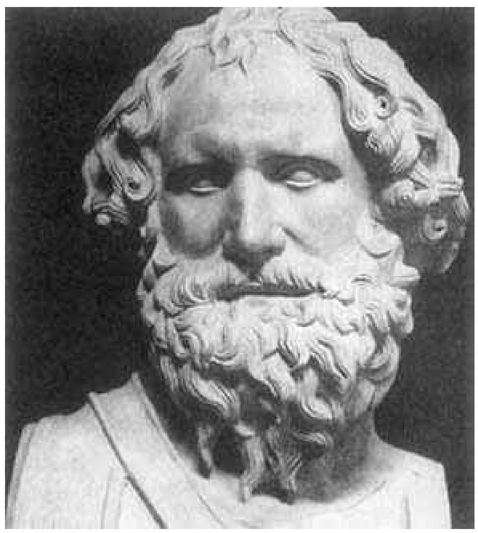
Note 4.7.B. We saw the quadratrix of Hippias, named for Hippias of Elis (circa 469 BCE–circa 400 BCE), in Section 4.6. Trisection of an Angle (see Note 4.6.E). The quadratrix can also be used to square the circle, though this construction is more complicated than the trisection of an angle. This is explained in Thomas Heath's *A History of Greek Mathematics, Volume I. From Thales to Euclid* (Clarendon Press, Oxford, 1921) on pages 227 and 228. It seems that Hippias was concerned with the trisection of the angle, and the squaring of the circle with the quadratrix is "presumably due" (as Heath puts it) to Dinostratus (circa 390 BCE–circa 320 BCE); the source for this claim is Pappus of Alexandria's (circa 290–circa 350) Book IV of *The Collection*.

Note 4.7.C. Archimedes of Syracuse (287 BCE–212 BCE) introduced the "Archimedean spiral" in his *On Spirals*, which has survived and is part of *The Works of Archimedes*, edited by Thomas Heath (Cambridge University Press, 1897). This is still in print by Dover publications; *On Spirals* appears on pages 151 to 188. On page 165, the following definition of Archimedes is given:

> "If a straight line drawn in a plane revolve at a uniform rate about one extremity which remains fixed and return to the position from which it started, and if, at the same time as the line revolves, a point move at a uniform rate along the straight line beginning from the extremity which remains fixed, the point will describe a *spiral* in the plane."

This definition is not rigorous by today's standards, due to the reference to the informal ideas of movement and time (though we would be remiss if we did not observe that Archimedes has snuck in an intuitive idea of a continuum here). Of course

Archimedes did not have access to Cartesian coordinates (due to René Descartes [March 31, 1596–February 11, 1650] who introduced them in his *La Géométrie* which appeared as a supplement to his *Discours de la méthode* in 1637) nor to polar coordinates. With polar coordinates (r, θ) we can simply express the Archimedean spiral as the function $r = a\theta$ where a is some constant. Notice that Archimedes definition of spiral limits the polar coordinate definition to $\theta \in [0, 2\pi]$, since he only considers one revolution ("return to the position from which it started").



MacTutor History of Mathematics Archive biography of Archimedes (accessed 5/9/2023)

We consider Archimedes in more depth in Section 6.2. Archimedes. The Archimedean spiral is addressed in the historical component of Introduction to Modern Geometry (MATH 4157/5157) in Section 4.2. The Archimedean Spiral. The book on which the historical information for Introduction to Modern Geometry is based is Alexander Ostermann and Gerhard Wanner's *Geometry by Its History*, Springer Verlag (Undergraduate Texts in Mathematics, 2012). The figure below is from this source. It illustrates how to trisect an angle using the Archimedean spiral. For the angle in standard position with point P on the Archimedean spiral and the terminal side of the angle, we simply trisect the terminal side. If, as shown in the figure, the terminal side has length r, then the point on the terminal side a distance $r/3$ from the origin determines a circle centered at the origin of radius $r/3$. This circle intersects the Archimedean spiral at a point that determines an angle 1/3 the size of the given angle.

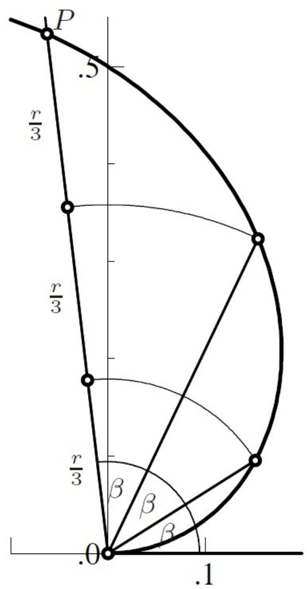


Fig. 4.4. Archimedean spiral for the trisection of an angle

Note. The Archimedean spiral makes it very easy to trisect (or even multisect) an angle, and it is quite likely that this is why it was introduced. It can also be used for the quadrature of the circle. In Figure 35, we have graphs of the Archimedean spiral $r = a\theta$ and the circle $r = a$ in polar coordinates. For a given angle θ, we know that the arc length around the circle is $a\theta$ and the distance from O to point P is also $a\theta$. Since we can construct a right angle $\pi/2$, with $\theta = \pi/2$ and P the corresponding point on the Archimedean spiral, we have that the length of line segment OP equals the arc length of 1/4 of the circumference of the circle. That is, we have $OP = (2\pi a)/4 = a\pi/2$ or $\pi = 2(OP)/a$. If a is constructible then $1/a$ is constructible and hence π can be constructed with the tools of compass, straight edge, *and* the Archimedean spiral. Recall that with a compass and straight edge, we may construct any real number that results from taking square roots a finite number of times and applying a finite number of field operations to elements of $\mathbb{Q}$ (see my online notes for Introduction to Modern Algebra 2 [MATH 4137/5137] on Section VI.32. Geometric Constructions; notice Theorem 32.6). The same idea holds here, but we are starting with the rationals and π (we are then in the "extension field $\mathbb{Q}[\pi]$"). This allows us to construct $\sqrt{\pi}$ and $a\sqrt{\pi}$, so that the $(a\sqrt{\pi}) \times (a\sqrt{\pi})$ square can by constructed, thus squaring the circle of radius a.

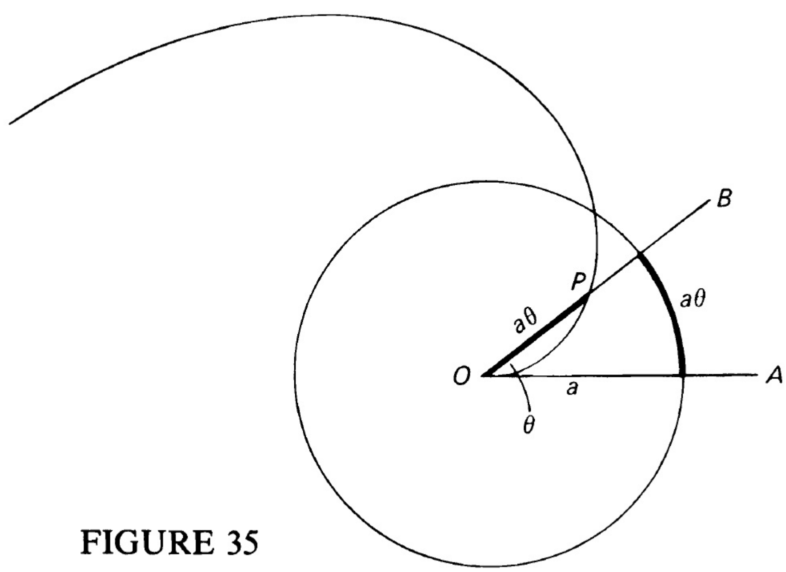


FIGURE 35