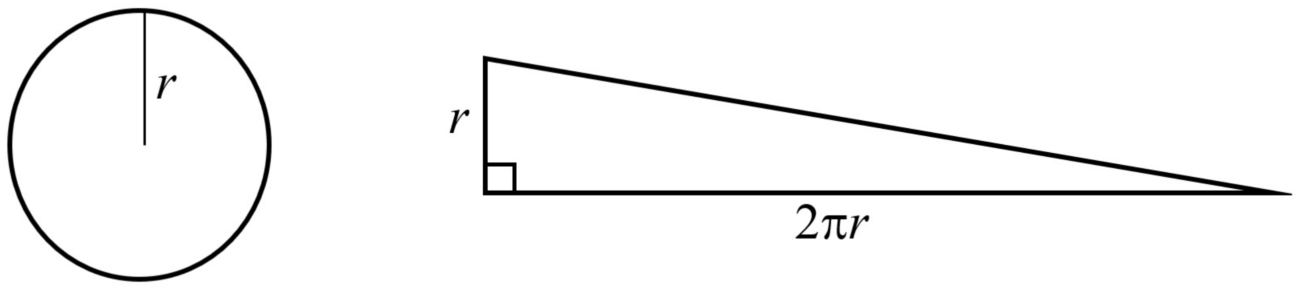
4.8. A Chronology of π

Note. Eves presents a brief history of π, starting with Archimedes and going up through the 1980s (and computations of decimals of π). Eves gives as a more detailed reference Herman Schepler's "The Chronology of Pi" which appears in a series of three papers in *Mathematics Magazine*: **23**(3), 165–170 (January/February 1950), **23**(4), 214–228 (March/April 1950), and **23**(5), 279–283 (May/June 1950). A more contemporary reference (and an easy read) is David Blatner's *The Joy of* π (Walker and Company, 1997). This short (130-odd page) book is written at the popular level. It includes lots of history and it has the gimmick of listing one million decimals of π. The associated website for the book is at: www.joyofpi.com (accessed 5/11/2023). A far more detailed reference if *Pi: A Source Book*, Third Edition, by Len Berggren, Jonathan Borwein, and Peter Borwein (Springer, 2004). This source has 70 chapters and runs over 700 pages.

Note. First, we address some of the weird history. Famously, in the King James version of the Bible (as Eves mentions in a footnote), 1 Kings 7:23 and 2 Chronicles 4:2 read roughly the same: "Also he made a molten sea of ten cubits from brim to brim, round in compass, and five cubits the height thereof; and a line of thirty cubits did compass it round about." This quote is the biblegateway.com version of 2 Chronicles 4:2. Literally interpreted, this implies the existence of a circle of radius $r = 5$ cubits and a circumference ("compass it round") of $C = 30$ cubits. Since, for a circle, $C = 2\pi r$ this implies that $\pi = 3$. Another well-known and related incident is the "Indiana Pi Bill." In 1897, the Indiana General Assembly considered bill

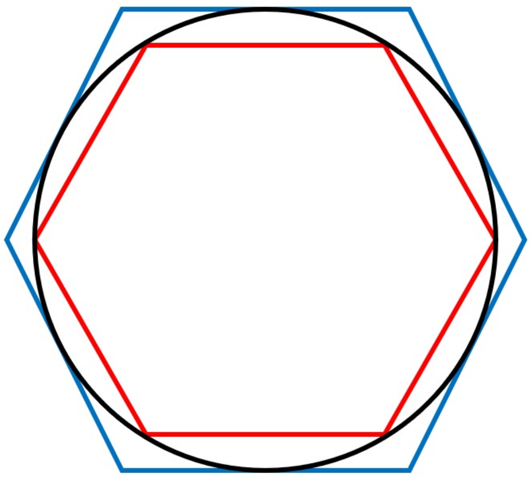
#246 which would have legally set a value of π at roughly 3.2 (not exactly at as 3, as is sometimes reported). The bill actually concerned the squaring of the circle (and some numerical rounding off). Bad publicity had been brought to the bill and it was indefinitely postponed (so it did not make it to a vote). For those interested, the Wikipedia page on the Indiana Pi Bill gives more details. Chapter 26 of *Pi: A Source Book* also gives details (and a copy of the bill). The webpages of this note were accessed 5/11/2023.

Note 4.8.A. We start our history with a definition of π. In Euclid's (circa 325 BCE–circa 265 BCE) *Elements*, Book XII, Proposition 2 it is shown that "Circles are to one another as the squares on the diameters." That is, the area of a circle is proportional to the square of its diameter or, equivalently, "the area of a circle is proportional to the square of its radius." The constant of proportionality between the area of a circle and the square of its radius is, by definition, π. So by is defined by the equation $A = \pi r^2$. We are also familiar with the formula for the circumference of a circle as $C = 2\pi r$. However, we need to establish this relationship since the parameter π is is defined in terms of the area of a circle, not its circumference. The is done in Archimedes' *Measurement of a Circle*. In Proposition 1 of this work, Archimedes (287 BCE–212 BCE) proves: "The area of any circle is equal to a right-angled triangle in which one of the sides about the right angle is equal to the radius, and the other to the circumference of the circle." This relates the circumference C, radius r, and area A of a circle as $A = \frac{1}{2}Cr$.

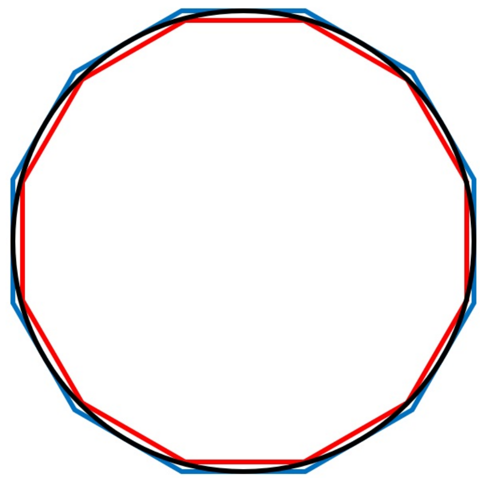


Since we have defined π by the equation $C = 2\pi r$, we now have $A = \frac{1}{2}(2\pi r)r = \pi r^2$. Archimedes proves his Proposition 1 with the method of exhaustion by first showing that the area of the circle cannot be greater than the area of the triangle. He assumes that the area of the circle is greater than the area of the triangle, then he inscribes a polygon in the circle that has an area greater than the area of the triangle (this can be done under the assumption), then he shows that the polygon *must* have an area less than the area of the triangle. This contradiction shows that the area of the circle cannot be greater than the area of the triangle. He next assumes that the area of the circle is less than the area of the triangle and similarly gets a contradiction based on circumscribed polygons. This is illustrated in my online PowerPoint presentation (and supplement to Section 6.2. Archimedes) Archimedes: 2,000 Year Ahead of His Time and the transcript of the presentation (in PDF).

Note 4.8.B. In Proposition 3 of *Measurement of a Circle* (which Eves dates to circa 240 BCE) Archimedes proves: "The ratio of the circumference of any circle to its diameter is less that $3\frac{1}{7}$ but greater than $3\frac{10}{71}$." Notice that $3\frac{1}{7} = 22/7$ and this is often used as a crude approximation of π. Rounded to five decimal places, we have $3\frac{1}{7} \approx 3.14286$, $\pi \approx 3.14159$, and $3\frac{10}{71} \approx 3.14085$. Archimedes starts with a circle of given radius, and inscribes and circumscribes it with regular hexagons. He then subdivides the hexagons to get regular dodecagons (i.e., 12 sided regular polygons). Notice in the figure below that even at this stage the dodecagons agree closely with the circle.

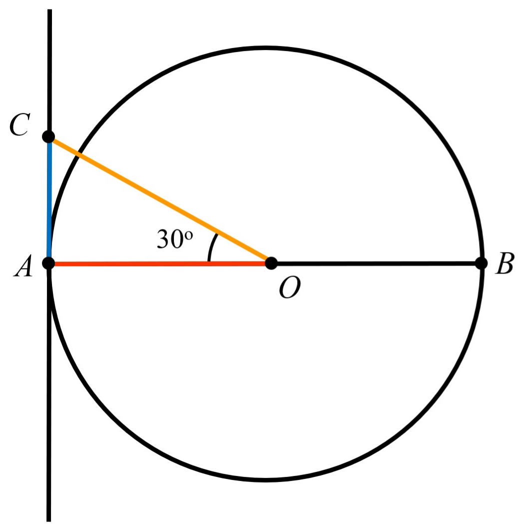


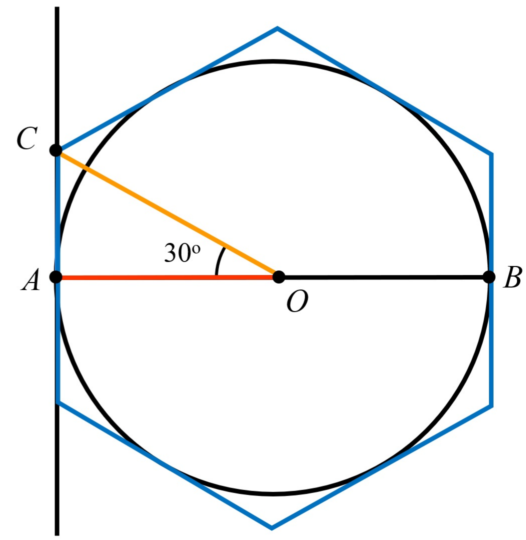
Inscribed Hexagon
Circumscribed Hexagon



Inscribed Dodecagon
Circumscribed Dodecagon

He then bisects the edges of the dodecagon another three times until he has produced an inscribing and circumscribing regular 96-gon. His estimation of the perimeters of the regular 96-gons is where the approximations or $3\,\frac{1}{7}$ and $3\,\frac{10}{71}$ come from. The detailed computations that Archimedes used are given in the previously mentioned supplement Archimedes: 2,000 Year Ahead of His Time (in PowerPoint) and the transcript of the presentation (in PDF). We go through the first two steps of Archimedes proof here, to give the flavor of his argument. Since Archimedes does not have access values of trigonometric functions, he must put bounds on the ratios of the sides of right triangles, starting with a 30-60-90 triangle as illustrated below. With O as the center of the triangle and A as a point on the circle, he takes a tangent to the circle at A and introduces a 30° angle $\angle AOC$, as illustrated.





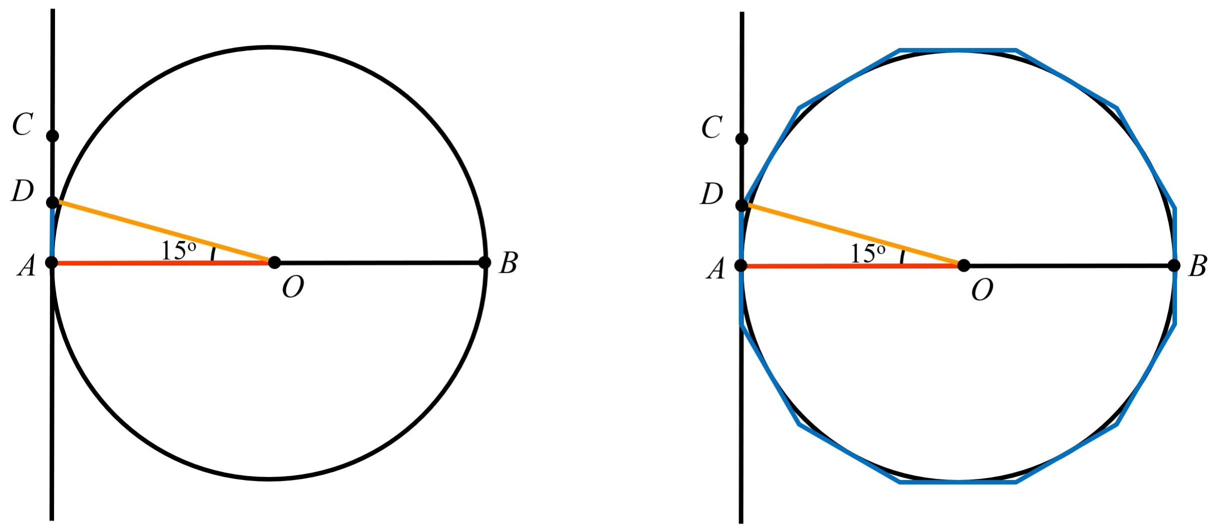
Since Archimedes needs the length of side AC in order to find the perimeter of the hexagon, he needs knowledge about the trig functions of $30°$. He will use $\cot 30° = \sqrt{3}$, so he needs an upper and lower bound on $\sqrt{3}$. Without explanation, he states that $\frac{265}{153} < \sqrt{3} < \frac{1351}{780}$ (it is straightforward to verify this). An explanation of the upper bound will be seen in Section 6.6. Heron; see Note 6.6.E. Then in triangle $\triangle AOC$, we have $OA/AC = \cot 30° = \sqrt{3} > 265/153$. Also $OC/AC = \csc 30° = 2 = 306/153$ (he is getting a common denominator here; we'll use this below). Since AC is $1/12$ of the perimeter of the circumscribed hexagon, we have that the circumference of the circle is bounded above by $12(AC) < 12(153(OA)/265) = 1836(OA)/265$. Since the radius of the circle is $r = (OA)$, then its circumference is $2\pi r = 2\pi(OA)$ and hence we have the upper bound on π of $918/265 \approx 3.46415$. Next, Archimedes bisects the $30°$ angle to get a $15°$ angle, from which he generates a dodecagon as shown in the next figure. Let B be a point on the line tangent to the circle at point A, so that angle $\angle AOC$ has measure $15°$. By Euclid's Proposition

VI.3, we have $(OC)/(OA) = (CD)/(AD)$. Therefore

$$\frac{(OC)}{(OA)} + 1 = \frac{CD)}{(AD)} + 1 \text{ or } \frac{(OC)+(OA)}{(OA)} = \frac{(CD)+(AD)}{(AD)} = \frac{(AC)}{(AD)}$$

and $\dfrac{(OC)+(OA)}{(AC)} = \dfrac{(OA)}{(AD)}$. From the computations for the circumscribed hexagon, now we have

$$\frac{(OA)}{(AD)} = \frac{(OC)+(OA)}{(AC)} = \frac{(OC)}{(AC)} + \frac{(OA)}{(AC)} = \sqrt{3} + 2 > \frac{265}{153} + \frac{306}{153} = \frac{571}{153}.$$



As with the circumscribed hexagon, the dodecagon gives an estimate of π. Since AD is 1/24 of the perimeter of the circumscribed dodecagon, we have that the circumference of the circle is bounded above by $24(AD) < 24(153(OA)/571) = 3672(OA)/571$. Since the radius of the circle is $r = (OA)$, then its circumference is $2\pi r = 2\pi(OA)$ and hence we have the upper bound on π of $1836/571 \approx 3.21541$ (an improvement over the estimate $918/265 \approx 3.46415$ based on the circumscribed hexagon). Archimedes applies this technique three more times and considers (1) a $7.5°$ angle and a 24-gon, (2) a $3.75°$ angle and a 48-gon, and (3) a $1.875°$ angle and a 96-gon. In the 96-gon he shows that the ratio of the radius of the circle to half the length of a side of the 96-gon is bounded below by $(4673\,\frac{1}{2})/153$. From this, he shows that

$$\pi < \frac{14688}{4673\,\frac{1}{2}} = 3 + \frac{667\,\frac{1}{2}}{4673\,\frac{1}{2}} < 3 + \frac{667\,\frac{1}{2}}{4672\,\frac{1}{2}} = 3\,\frac{1}{7}.$$

For a lower bound on π, Archimedes again start with a hexagon (this time inscribed in the circle). With a very similar technique he shows $\pi > \dfrac{6336}{2017\,\frac{1}{4}} > 3\,\dfrac{10}{71}$. Hence, he has his bounds on the value of π: $3\,\frac{10}{71} < \pi < 3\,\frac{1}{7}$. Archimedes technique of estimating π using inscribed and circumscribed polygons is the *classical method*.

Note. Claudius Ptolemy (circa 85 CE–circa 165 CE) is best known for his *Almagest* (originally title *The Mathematical Compilation*). In this, he laid out his Earth-centered (i.e., geocentric) model of the solar system, with the planets traveling on epicycles to explain the occasional retrograde motion. Using formulae analogous to out formulae for $\sin(a+b)$ and $\sin(a-b)$, Ptolemy gave estimates of the length of a chord on a circle corresponding to a central angle of all sizes a multiple of 1/2 degree.

Image from MacTutor History of Mathematics Archive biography of Ptolemy (accessed 5/13/2023)

This allowed him to circumscribe a 360-gon in a circle from which he got an estimate of π of $3\,\frac{17}{120} \approx 3.14167$. In fact, he gave his estimation base 60 (i.e., sexagesimal) as $3\,8'\,30''$ For more details on the early history of astronomy, see my online notes for Astronomy (PS 215 at Auburn University) on Chapter 3. Early Astronomy. For more details on the chord function and Ptolemy's estimation of π, see my online notes for the history component of Introduction to Modern Geometry (MATH 4157/5157) on Section 5.1. Ptolemy and the Chord Function.

Note. We now give Eves' quick list of the next 1,000 years of relevant work.

Zu Chongzhi (or "Tsu Ch'ung-chih," 429 CE–501 CE)

Zu Chongzhi, or "Tsu Ch'ung-chih" as Eves spells it, (429 CE–501 CE) gave the rational approximation to π of $355/113 \approx 3.1415929$ (accurate to 6 decimal places) in his text *Zhui shu* (Method of Interpolation). This book did not survive, but its results are given in the 7th century *History of the Sui dynasty* by Li Chunfeng (602–670). Since Chongzhi's book is lost, we can only speculate as to the details of his argument. Chunfeng gives some information and reports that by considering a circle of diameter 10,000,000 Chongzhi found the circumference to be less than

31,415,927 and greater than 31,415,926. Since the circumference is π times the diameter, then he knows that $3.1415926 < \pi < 3.1415927$. Based on this, he deduced his rational approximation 355/113. This historical information and the image above are from the MacTutor History of Mathematics Archive biography of Zu Chongzhi (accessed 5/17/2023).

Note. The Indian mathematician Aryabhata the Elder (or "Āryabhata," as Eves spells it, or Aryabhata I) gave an approximation of π of

$$62{,}832/20{,}000 = 31{,}416/10{,}000 = 3927/1250 = 3.1416,$$

though he did not explain his technique. His results appear in his *Aryabhatiya* (which he finished in 499), though no explanation is given.

Aryabhata the Elder (476–550). This statue is at the Inter-University Centre for Astronomy and Astrophysics in Pune, India.

Eves speculates that the result may have come from an earlier Greek source of from the perimeter of a regular 384-gon inscribed in a circle (see Eves' page 118). The *Aryabhatiya* has 118 verses, 33 of which summarize Hindu mathematics at the time. Sixty-six mathematical rules are stated, but without proof. It covers arithmetic, algebra, continued fractions, plane trigonometry and spherical trigonometry (including a table of sines). Twenty-five of the verses involve measurements of time and planetary motion. This historical information and the image below are from the MacTutor History of Mathematics Archive biography of Aryabhata the Elder (accessed 5/17/2023).

Indian mathematician Bhaskara (Bhāskara, Bhaskara II, or Bhaskaracharya [“Bhaskara the Teacher”]; 1114–1185)

Revised: 5/11/2024