5.4. Content of the "Elements"

Note. In this section we consider the material of the thirteen books of Euclid's *Elements*. We put emphasis on Book I. "American high-school plan and solid geometry texts contain much of the material found in Books I, III, IV, XI, and XII" (Eves, page 144). An alternative presentation of this material is given in my online notes for the history component of Introduction to Modern Geometry (MATH 4157/5157); see Chapter 2 (the notes for this section duplicate some of that material). However, the *Elements* is not exclusively about geometry, and it also covers number theory and elementary geometric algebra. The thirteen books contain a total of 465 propositions (i.e., "theorems"). Broadly, Books I–VI deal with plane geometry, Books VII–X with arithmetic, and Books XI–XIII cover solid geometry (including the construction of the five famous platonic solids). The size of the books varies between about 2.5% of the whole for the smallest, Book II, and 25% for Book X. Each of the others is roughly 5–8% of the total.

Note 5.4.A. Book I starts unceremoniously with 23 definitions (and no pictures). As a sampling of these definitions, we have:

Definition 1. A point is that which has no part.

Definition 4. A straight line is a line which lies evenly with the points on itself.

Definition 8. A plane angle is the inclination to one another of two straight lines in a plane which meet one another and do not lie in a straight line.

Definition 23. Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

Of course this raises as many questions as it answers, since we now focus on the terms "part," "lies evenly," "inclination," and the meaning of "being produced indefinitely in both directions." Since we can only define new terms using old terms, at some point we must stop and simply take certain terms as undefined. The properties of these undefined terms are given to them by the postulates. For more explanation of these ideas, see my online notes for Introduction to Modern Geometry (MATH 4157/5157) on Section 1.3. Axiomatic Systems.

Note 5.4.B. Book I includes five postulates (or "assumptions"). They are:

Postulate 1. To draw a straight line from any point to any point.

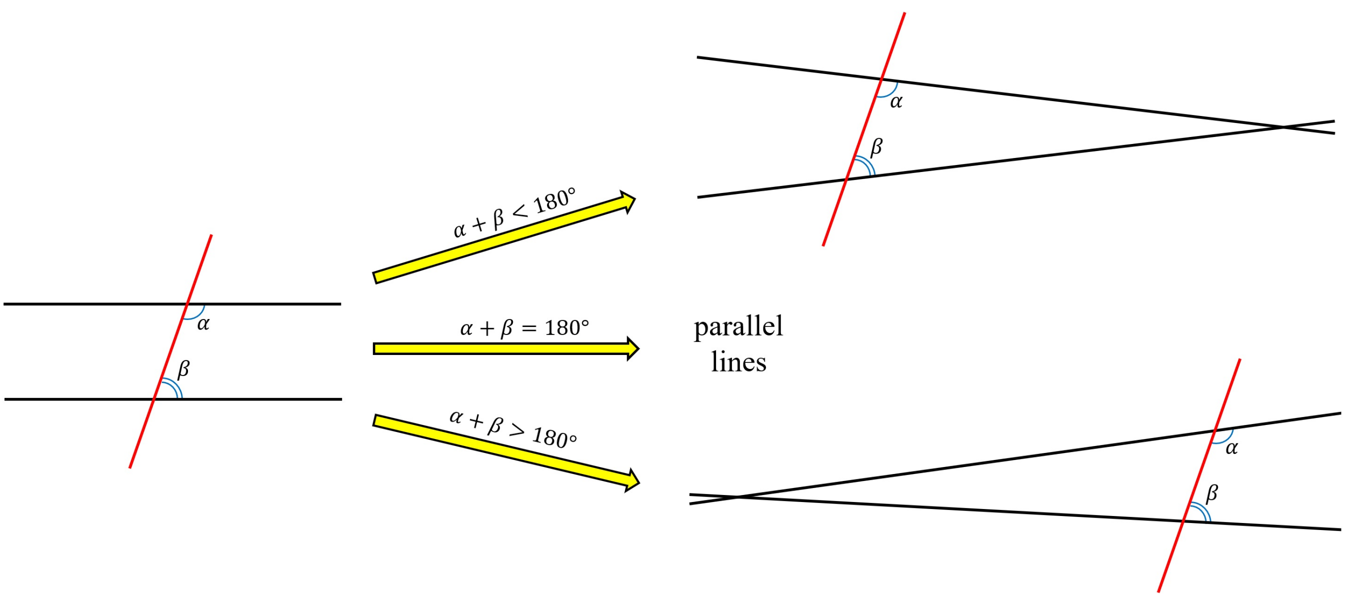
Postulate 2. To produce a finite straight line continuously in a straight line.

Postulate 3. To describe a circle with any center and radius.

Postulate 4. That all right angles equal one another.

Postulate 5. That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles. (We'll deal in more detail with definitions and assumptions of the *Elements* in Section 5.7. Formal Aspect of the "Elements".) The first three postulates are meant to insure the existence of certain constructions. Postulate 1 means that if two (distinct) points are given, then a line containing those two points can be constructed. Postulate 2 means that if a line segment (a "finite straight line") is given, then it can be extended to a (infinite, unbounded) line. Postulate 3 means that if a point is given and if a distance is given (in terms a particular line segment), then a circle with the point as its center and the distance as its radius can be constructed. This terminology is used throughout the Elements (along with an unusual way of distinguishing between lines and line segments). We described in Section 4.4. The Euclidean Tools how the first three postulates are inspired by compass and straight edge constructions. Postulate 4 claims an equality of a certain class of angles; it is actually the measure of the angles that are being claimed to be equal (though the measure of an angle is never defined). Notice that the first four postulates are unsurprising and uncomplicated. However, Postulate 5 could use some additional exploration.

Note 5.4.C. Postulate 5 is the Parallel Postulate. Think of the "two straight lines" as being given, and then a "straight line falling" on these as a transversal cutting both lines. The following figure illustrates the Euclid's version of the Parallel Postulate:



The idea of "interior angles" requires some concept of "betweenness." The condition "less than two right angles" requires (again) the idea of a measure of an angle (and, if we are being picky, the "side" of a line is never defined). So by our 21st century standards, Euclid lacks some rigor. A modern approach to an axiomatic development of Euclidean geometry is given in the axiomatic part of Introduction to Modern Geometry (MATH 4157/5157). Recall from Section 5.3. Euclid's "Elements" that Euclid's Parallel Postulate can be replace with Playfair's Axiom, "Given a line an a point not on the line, it is possible to draw exactly one line through the given point parallel to the line." (See Note 5.3.I.) This is easier to illustrate and is a common way to address the Parallel Postulate in high school geometry. In your humble instructor's high school math book *Modern School Mathematics: Geometry*, by Ray C. Jurgensen, Alfred J. Donnelly, Mary P. Dolciani, (Boston: Houghton Mifflin Company, 1969 and 1972), the Parallel Postulate is stated as a simplified version of Euclid's (see my online notes [a work in progress] on Chapter 5. Parallel Lines and Planes): "If two line are cut be a transversal so that corresponding angles are congruent, the lines are parallel." "Corresponding angles" in the figure above are α and the complement of β, $180° - \beta$, so these being equal implies $\alpha + \beta = 180°$, as in the center part of the figure above. Playfair's *Theorem* then follows from two theorems whose proofs are based on this version of the Parallel Postulate (Theorem 5-4 and Theorem 5-5 in the Jurgensen, Donnelly, and Dolciani book).

Note. Book I also contains five "Common Notions." These are related to arithmetic relationships concerning equality and "greater than." The common notions are:

Common Notion 1. Things which equal the same thing also equal one another.

Common Notion 2. If equals are added to equals, then the wholes are equal.

Common Notion 3. If equals are subtracted from equals, then the remainders are equal.

Common Notion 4. Things which coincide with one another equal one another.

Common Notion 5. The whole is greater than the part.

Note. Book I consists of 48 propositions that fall into three groups. The first 26 deal with properties of triangles and congruence. Propositions I.27 through I.32 deal with the theory of parallels. The remaining 16 propositions deal with parallelograms, triangles, and squares. Proposition I.47 is the Pythagorean Theorem and the last proposition of Book I, Proposition I.48, is its converse. Eves states (see page 145): "The material of this book was developed by the early Pythagoreans." We have seen that the Parallel Postulate stands out from the other postulates (and common notions). There is some circumstantial evidence that Euclid wanted to avoid using it as long as possible. The first 28 propositions of Book I do not require it, and Proposition 29 is the first to use the Parallel Postulate. The first 28 Postulates (with a few exceptions; see my online presentation on Euclidean Geometry and the discussion about Proposition 16) make up "neutral geometry." Neutral geometry consists of geometric properties that hold both in Euclidean geometry (in which the Parallel Postulate holds) and in non-Euclidean geometry (in which the Parallel Postulate is false).

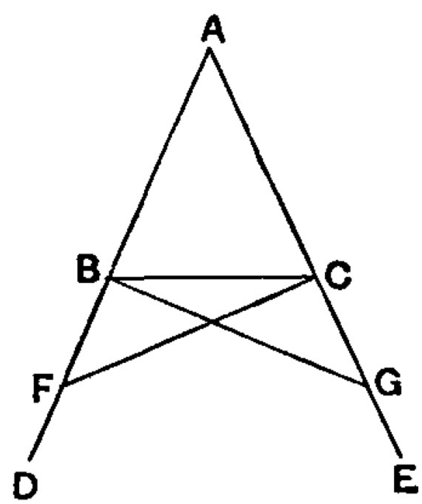
Note 5.4.D. Proposition I.4 establishes the congruence of two triangles such that they have two equal sides and the corresponding included angles equal (that is, side-angle-side, SAS). The proof follows by "applying" one triangle "to" the other. That is, by "superposition." In other words, one triangle is *transformed* to the other. As we'll see in Section 15.1. Logical Shortcomings of Euclid's "Elements", objections would come to the use of superposition. In fact, the idea of rigidly transforming a set of points in a geometric space from one position to another, is a fundamental concept in the modern idea of *transformational geometry*. In fact, ETSU had the graduate-level class Axiomatic and Transformational Geometry (MATH 5330) listed in the catalog until 2015. I have online notes for this class on the transformational geometry part of the class. Both plane Euclidean geometry (in Chapter V, "Mappings of the Euclidean Plane" of those notes) and plane hyperbolic geometry (in Chapter VI, "Mappings of the Inversive Plane") are covered in detail. The projective plane is covered in Chapter VII, "The Projective Plane and Projective Spaces." A brief version of the use of transformations in the complex plane related to hyperbolic geometry is given in my online notes on Supplemental Notes on III.3. Analytic Functions as Mappings: Möbius Transformations—with Supplemental Material from Hitchman's *Geometry with an Introduction to Cosmic Topology* (which is used in Complex Analysis 1, MATH 5510).

Note 5.4.E. Proposition I.5 states that the base angles of an isosceles triangle are equal. As stated in Heath's translation: "In isosceles triangles the angles at the base equal one another, and, if the equal straight lines are produced further, then the angles under the base equal one another." Eves states (page 145):

> "...it is said that many beginners found the proof so confusing that they abandoned further study of geometry. The proposition has been dubbed the *pons asinorum* ["ass' bridge"], or 'bridge of fools,' because of the fancied resemblance of the figure of the proposition [below] to a simple trestle bridge too steep for some novices to pass over"

According to the Merriam-Webster Dictionary online (accessed 7/24/2023), the first known use of the term *pons asinorum* was 1645. Euclid's proof is along he following lines.

> Let the isosceles triangle be $\triangle BAC$ with equal sides AB and AC. Produce sides AB and AC to points F and G, respectively, such that BF and CG are the same lengths. Draw line segments BG and CF. Then by Proposition I.4 the triangles $\triangle AFC$ and $\triangle AGB$ are congruent (since sides AF and AG are equal by construction, sides AB and AC are equal by hypothesis, and the included angles $\angle BAG$ and $\angle CAF$ are the same).

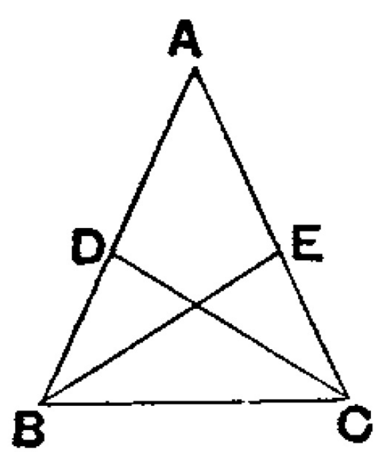


From page 251 of Heath's translation of the *Elements*, Volume 1

> Therefore sides BG and FC are equal, and angles $\angle BFC$ and $\angle CGB$ are equal. So, again by Proposition I.4, triangles $\triangle BFC$ and $\triangle CGB$ are congruent. Hence angles $\angle FBC$ and $\angle GCB$ are equal. We then have (by Common Notion 3) that $\angle ABC$ and $\angle ACB$ are equal, as claimed. *Q.E.D.*

Note 5.4.F. Proposition I.6 is the converse of Proposition I.5. That is, if two angles in a triangle are equal then the sides opposite them are equal. We outline Euclid's proof of this result as well, since it is the first time in the *Elements* that he uses the proof technique *reductio as absurdum* (or "proof by contradiction"). This technique is frequently used throughout the remainder of the *Elements*.

> Consider triangle $\triangle ABC$ where angles $\angle ABC$ and $\angle ACB$ are equal. ASSUME that $AB > AC$. Then there is a point D on AB such that $AC = DB$. By Proposition I.4, triangles $\triangle CBD$ and $\triangle BCA$ are congruent (Since $AC = DB$, $CB = BC$, and the included angle $\angle DBC$ is shared by both triangles).



From page 255 of Heath's translation of the *Elements*, Volume 1

But $\triangle CBD$ is a proper part of $\triangle BCA$, so the congruence gives a CONTRADICTION. So the assumption that $AB > AC$ is false, and we must have $AB \leq AC$. Similarly, we can assume that $AB < AC$ and get a contradiction so that we also have $AB \geq AC$. Therefore, $AB = AC$, as claimed. *Q.E.D.*

Note 5.4.G. Book II is much shorter than Book I, only containing two definitions fourteen propositions. It contains what we have called "geometric algebra." As described in Section 3.6. Algebraic Identities, Propositions II.4, II.5, and II.6 establish the following algebraic identities, respectively:

$$(a+b)^2 = a^2 + 2ab + b^2, \quad (a+b)(a-b) = a^2 - b^2, \quad 4ab + (a-b)^2 = (a+b)^2.$$

The results of this book were likely known to the Phythagoreans. Propositions II.12 and II.13 establish the Law of Cosines (without any reference to "cosines," of course). Obtuse-angled triangles are dealt with in Proposition II.12 and acute-angled triangles are dealt with in Proposition II.13. Eves paraphrases these as (see page 147): "In an obtuse-angled (acute-angled) triangle, the square of the side opposite the obtuse (acute) angle is equal to the sum of the squares of the other two sides increased (decreased) by twice the product of one of these sides and the projection of the other on it." Recall that the Law of Cosines states (with the obvious notation): $c^2 = a^2 + b^2 - 2ab\cos C$. The cosine function deals with the "projection" comment, and the facts that the cosine of an obtuse angle is negative and of an acute angle is positive deals with the "increased (decreased)" part of the statement.

Note 5.4.H. Book III covers properties of circles, and angles and lines (chords and tangents) associated with them. It consists of definitions of eleven definitions. It introduces the terminology of a line that "touches" a circle, by which is meant a tangent line to the circle. A *circle*, its *center*, and its *diameter* are defined in Book I in Definitions 15, 16, and 17, respectively. Some of the propositions include:

Proposition III.2. If two points are taken at random on the circumference of a circle, then the straight line joining the points falls within the circle. [In modern terminology, that is, a circle is a *convex set*.]

Proposition III.10. A circle does not cut a circle at more than two points.

Proposition III.13. A circle does not touch another circle at more than one point whether it touches it internally or externally. [That is, two circles can have at most one common point of tangency.]

Proposition III.17. From a given point to draw a straight line touching a given circle. [That is, a tangent line to a circle through a given point is constructed, ultimately based on Euclidean tools.]

Proposition III.27. In equal circles angles standing on equal circumferences equal one another whether they stand at the centers or at the circumferences. [By "equal circumferences" is meant equal arc lengths on the circles. So the claim is that central angles (with their vertex at the center of the circle containing the angle) or inscribed angles (with their vertex on the circle containing the angle) with equal arc lengths in equal circles are equal.]

Proposition III.30. To bisect a given circumference. [That is, for a given arc length on a circle to bisect the arc length.]

The results of Book III are contained in highschool geometry texts. The geometry of the circle contained in Book III (and in parts of Book IV) is not found in the work of the Pythagoreans, and is likely due to those who researched the three compass and straight edge constructions mentioned in Section 4.3. The Three Famous Problems.

Note 5.4.I. Book IV contains seven definitions and sixteen propositions. If covers straight edge and compass constructions of regular polygons with three, four, give, six, and fifteen sides, as well as the inscribing of these polygons within a given circle, and the circumscription about a given circle. Some of the propositions include:

Propositions IV.2, IV.3, IV.4 and IV.5. To inscribe in a given circle a triangle equiangular with a given triangle. To circumscribe about a given circle a triangle equiangular with a given triangle. To inscribe a circle in a given triangle. To circumscribe a circle about a given triangle.

Propositions IV.6, IV.7, IV.8, and IV.9. To inscribe a square in a given circle. To circumscribe a square about a given circle. To inscribe a circle in a given square. To circumscribe a circle about a given square.

Propositions IV.11, IV.12, IV.13, and IV.14. To inscribe an equilateral and equiangular pentagon in a given circle. To circumscribe an equilateral and equiangular pentagon about a given circle. To inscribe a circle in a given equilateral and equiangular pentagon. To circumscribe a circle about a given equilateral and equiangular pentagon.

Propositions IV.15 and IV.16. To inscribe an equilateral and equiangular hexagon in a given circle. To inscribe an equilateral and equiangular fifteen-angled figure in a given circle.

Note 5.4.J. Book V contains eighteen definitions and twenty-five propositions. This contains Eudoxus' (408 BCE–355 BCE) theory of proportion, which we first mentioned in Section 3.5. Discovery of Irrational Magnitudes; see Note 5.3.E. This theory is "applicable to incommensurable as well as commensurable magnitudes, [and] resolved the 'logical scandal' created by the Pythagorean discovery of irrational numbers" (Eves, pages 147 and 148). The "Eudoxian definition" of proportion (or equality of two ratios) is given as Definition 5 of Book V:

> **Definition 5.** "Magnitudes are said to be *in the same ratio*, the first to the second and the third to the fourth, when, if any equimultiples whatever are taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order."

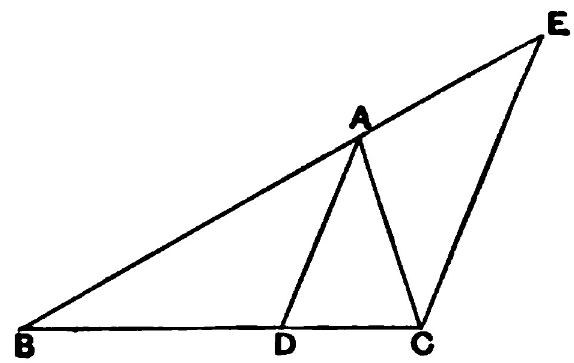
That is, for magnitudes A and B of the same kind (both line segments, or angles, or areas, or volumes; we are taking magnitudes to be positive here) and magnitudes C and D of the same kind, we have that $A/B = C/D$ by Definition 5 means that for all positive integers m and n: (1) $mA \leq nB$ (i.e., $A/B \leq n/m$) if and only if $mC \leq nD$ (i.e., $C/D \leq n/m$), and (2) $mA \geq nB$ (i.e., $A/B \geq n/m$) if and only if $mC \geq nD$ (i.e., $C/D \geq n/m$). Notice that the definition needs to be in terms of *products* of magnitudes which are already defined; the parenthetic comments here involving quotients is just for clarity (though quotients of integers is already established). This is similar to Richard Dedekind's (October 6, 1831–February 12, 1916) approach to irrationals in his *Dedekind cuts* and completeness of the real numbers. This is explained in my online Calculus 1 notes on Appendix A.6. Theory of the Real Numbers. You will also discuss the completeness of the real numbers in Analysis 1 (MATH 4217/5217) on Section 1.3. The Completeness Axiom. The propositions of Book V are largely just an elementary theory of arithmetic extended to include magnitudes. For example, Propositions V.1, V.2, V.3, V.5, and V.6 establish distribution and associativity properties of numbers and magnitudes. The remaining propositions develop the theory of ratios and proportions.

Note 5.4.K. Book VI consists of four definitions and thirty-three propositions. It applies the Eudoxian theory of proportions of Book V to plane geometric figures. Proposition VI.1 is the basis for the entire of Book VI except the last proposition VI.33 (according to David Joyce's online copy of Book VI (accessed 7/26/2023).

Proposition VI.1. Triangles and parallelograms which are under the same height are to one another as their bases. [The "are to one another as their bases" means that they have areas that are in the same proportion as the lengths of their bases. We give three proofs of this in Section 5.5. The Theory of Proportion.]

Some other propositions of Book VI include:

Proposition VI.3. If an angle of a triangle is bisected by a straight line cutting the base, then the segments of the base have the same ratio as the remaining sides of the triangle; and, if segments of the base have the same ratio as the remaining sides of the triangle, then the straight line joining the vertex to the point of section bisects the angle of the triangle. [Consider the following figure from Heath's translation of the *Elements*:



This proposition means that, in triangle $\triangle ABC$, angle $\angle BAC$ is bisected if and only if the ratio of BD to DC is the same as the ratio of AB to AC. Point E is introduced for use in the proof.]

Propositions IV.4 and IV.5 imply that triangle are similar if and only if they have equal corresponding angles ("Angle-Angle-Angle," or AAA). Several of the other propositions concern similar triangles. Proposition VI.11 concerns the construction of a "third mean proportional" of two line (segments). If a and b are magnitudes, then their third mean proportional is magnitude c such that $a : b = b : c$ (or $a/b = b/c$). Proposition VI.12 concerns the construction of a "fourth mean proportional" of three line (segments). If a, b, and c are magnitudes, then their fourth mean proportional is magnitude d such that $a : b = c : d$ (or $a/b = b/c$). Proposition VI.13 concerns the construction of a "mean proportional" (or *geometric mean*) of two line (segments). If a and b are magnitudes, then their mean proportional is x such that $a : x = x : b$ (or $ab = x^2$). Proposition VI.18 concerns the construction of a rectilinear figure (that is, a polygon) similar to a given rectilinear figure and a given line segment (the line segment being required to correspond to some edge of the rectilinear figure, thus setting up the proportion between the the given figure and the one to be constructed). Proposition VI.30 concerns cutting a line segment in "extreme and mean ratio." This means taking a given line segment and cutting it into pieces of lengths a and b such that $(a + b) : b = b : a$ (in which case one can make a rectangle with dimensions a and $a + b$ which is equal in area to a square of dimension b). This is used in the construction of a pentagonal face of a regular dodecahedron in Proposition 17 of Book XIII. Book VI also contains several results concerning parallelograms. We mention one additional proposition form this book, which is a generalization of the Pythagorean Theorem.

Proposition VI.31. In right-angled triangles the figure on the side opposite the right angle equals the sum of the similar and similarly described figures on the sides containing the right angle.

This means that instead of drawing squares on the sides of the right triangle (to get the familiar "$a^2 + b^2 = c^2$"), we can draw any similar figures which are in the same proportions as are the sides of the right triangle. Euclid's proof in Book VI involves rectilinear figures. However, this is known to hold for other shapes, as demonstrated by the lunes of Hippocrates. These were mentioned in passing in Section 4.7. Quadrature of the Circle, and mentioned in more detail in the historical component of Introduction to Modern Geometry (MATH 4157/5157) in Section 1.8. Three Famous Problems of Greek Geometry (see Figure 1.23(c) in those notes). Eves concludes that the Pythagoreans probably knew many of the results of Book VI: "There probably is no theorem in this [Book VI] that was not known to the early Pythagoreans, but the pre-Eudoxian proofs of many of them were at fault, since they were based upon the incomplete theory of proportion" (Eves, page 148).

Note 5.4.L. Books VII, VIII, and IX deal with elementary number theory. ETSU has a class (rarely offered these day; that is, in the 2020s) on this very topic, Elementary Number Theory (MATH 3120). I have online notes for Elementary Number Theory, which give a modern approach to the topics. Book VII of the *Elements* has 22 definitions (including such things as number [by which is meant a positive integer], multiple, even number, odd number, prime number, relatively prime numbers, square number, cube number, and a perfect number) and 39 propositions. The first is proposition is the Euclidean Algorithm:

Proposition VII.1. When two unequal numbers are set out, and the less is continually subtracted in turn from the greater, if the number which is left never measures the one before it until a unit is left, then the original numbers are relatively prime.

By "measures," Euclid means "divides." The statement of this in Elementary Number Theory (MATH 3120) is:

Theorem. The Euclidean Algorithm.

If a and b are positive integers, $b \neq 0$, and

$$\begin{array}{ll} a = bq + r, & 0 \le r < b, \\ b = rq_1 + r_1, & 0 \le r_1 < r, \\ r = r_1 q_2 + r_2, & 0 \le r_2 < r_1, \\ \vdots & \vdots \\ r_k = r_{k+1} q_{k+2} + r_{k+2}, & 0 \le r_{k+2} < r_{k+1}, \end{array}$$

then for k large enough, say $k = t$, we have $r_{t-1} = r_t q_{t+1}$, and the greatest common divisor of a and b is r_t.

Proposition VII.2 involves finding the greatest common divisor of two numbers, so we see that this can be done with the Euclidean Algorithm. Some other elementary arithmetic properties include: $ab = ba$ (Commutativity of Multiplication, Proposition VII.16), $b : c = ab : ac$ (Proposition VII.17), and $a : b = c : d$ if and only if $ad = bc$ (Cross Multiplication, Proposition VII.19). A related result is the following:

Proposition VII.21. Numbers relatively prime are the least of those which have the same ratio with them.

This allows us to express a fraction "in lowest terms" (the converse if given in Proposition VII.22). Book VII contains several results on prime numbers, including Proposition VII.31 which states that any composite number is divisible by some prime number. These are first steps in the direction of the Fundamental Theorem of Arithmetic, which will be realized in Book IX.

Note 5.4.M. Book VIII has no knew definitions, but contains twenty-seven propositions. It is concerned largely with continued proportions and related geometric progressions. As examples, Proposition VIII.2 concerns the construction of a sequence of the form $a^{n-1}, a^{n-2}b, a^{n-3}b^2, \ldots, a^1b^{n-2}, b^{n-1}$ where the ratio in lowest terms is $a : b$. Proposition VIII.14 states that if c^2 divides d^2, then c divides d. Proposition VIII.15 is a similar result for c^3 and d^3.

Note 5.4.N. Book IX has no new definitions, but contains thirty-six propositions, some of them rather significant. We present a few of these.

Proposition IV.14. If a number is the least that is measured by prime numbers, then it is not measured by any other prime number except those originally measuring it.

This is the Fundamental Theorem of Arithmetic! As stated in Elementary Number Theory (MATH 3120), the theorem is (see Section 2. Unique Factorization, Theorem 2.2):

The Fundamental Theorem of Arithmetic (or **The Unique Factorization Theorem**).

Any positive integer greater than 1 can be written as a product of primes in one and only one way.

A proof is given in my online notes for Elementary Number Theory (MATH 3120).

"Euclid's proof of IX.20 (*the number of prime numbers is infinite*) has been universally regarded by mathematicians as a model of mathematical elegance" (Eves, page 148). Euclid states it as:

Proposition IX.20. Prime numbers are more than any assigned multitude of prime numbers.

The proof as given in Elementary Number Theory (MATH 3120) is as follows.

> We give a proof by contradiction. ASSUME there are only finitely many primes, say $p_1, p_2, \ldots, p_r$. Consider the integer $n = p_1p_2 \cdots p_r + 1$. By Proposition VII.32, n is divisible by a prime and since we have assumed there are only finitely many primes, the divisor must be one of $p_1, p_2, \ldots, p_r$. Suppose it is p_k.

Then we have p_k divides n and p divides $p_1 p_2 \cdots p_r$. So, by Proposition VII.7, p_k divides $n - p_1 p_2 \cdots p_r = 1$. But this is a CONTRADICTION since nor prime divides 1. Therefore the assumption that there are finitely many primes must be false and hence there are infinitely many primes, as claimed. *Q.E.D.*

Proposition IX.35 deals with a sum of a geometric sequence (though this is not clear from the statement):

Proposition IX.35. If as many numbers as we please are in continued proportion, and there is subtracted from the second and the last numbers equal to the first, then the excess of the second is to the first as the excess of the last is to the sum of all those before it.

With the numbers as $a_1, a_2, \ldots, a_n, a_{n+1}$, the "continued proportion" hypothesis means $a_1 : a_2 = a_2 : a_3 = \cdots = a_n : a_{n+1}$. If the ratio is r, then the claim is that

$$a + ar + ar^2 + \cdots + ar^{n-1} = a\frac{r^n - 1}{r - 1}.$$

You may see a proof of this in Calculus 2 (MATH 1920) or Mathematical Reasoning (MATH 3000). Euclid gives a geometric argument. In Section 3.3. Pythagorean Arithmetic, we defined a positive integer as *perfect* if it is the sum of its proper divisors. We related perfect numbers and Mersenne primes: If for some $n > 1$ we have $2^n - 1$ prime (called a Mersenne prime), then $2^{n-1}(2^n - 1)$ is a perfect number. See Note 3.3.C. Euclid states this result as follows:

Proposition IX.36. If as many numbers as we please beginning from a unit are set out continuously in double proportion until the sum of all becomes prime, and if the sum multiplied into the last makes some number, then the product is perfect.

The "double proportion" part corresponds to the powers of 2. The "sum of all" (by Proposition IX.35 with $a = 1$ and $r = 1$) is $1 + 2 + 2^2 + \cdots + 2^{n-1} = 2^n - 1$, so that this being prime means that we have a Mersenne prime. With "the sum multiplied by the last one" we have that $2^{n-1}(2^n - 1)$ is perfect.

Note 5.4.O. Book X has sixteen definitions and 115 propositions. As mentioned at the beginning of this section of notes, Book X makes up 25% of the *Elements*. Eves says (page 149): "Book X deals with irrationals—that is, with line segments that are incommensurable with respect to some given line segment. Many scholars regard this book as perhaps the most remarkable book in the *Elements*." As observed in Supplement. Proclus's Commentary on Eudemus' History of Geometry, it is the theory of irrationals of Theaetetus of Athens (circa 417 BCE–circa 369 BCE) that makes up Book X, "...but the extraordinary completeness, elaborate classification, and finish are usually credited to Euclid" (Eves, page 149). We now go through the first four definitions, which set the stage for Book X.

Definition 1. Those magnitudes are said to be *commensurable* which are measured by the same measure, and those *incommensurable* which cannot have any common measure.

Definition 2. Straight lines are *commensurable in square* when the squares on them are measured by the same area, and *incommensurable in square* when the squares on them cannot possibly have any area as a common measure.

Definition 3. With these hypotheses, it is proved that there exist straight lines infinite in multitude which are commensurable and incommensurable respectively, some in length only, and others in square also, with an assigned straight line. Let then the assigned straight line be called *rational*, and those straight lines which are commensurable with it, whether in length and in square, or in square only, *rational*, but those that are incommensurable with it *irrational*.

Definition 4. And the let the square on the assigned straight line be called *rational*, and those areas which are commensurable with it *rational*, but those which are incommensurable with it *irrational*, and the straight lines which produce them *irrational*, that is, in case the areas are squares, the sides themselves, but in case they are any other rectilineal figures, the straight lines on which are described squares equal to them.

First, recall that by "magnitude" Euclid means an "amount" of something, such as (the length of) line segments, or angles, or areas, or volumes (see Note 5.4.J above). Definition 1 means that two magnitudes A and B (of the same kind) are *commensurable* if there is another magnitude C (of the same kind) such that A and B are both (natural number) multiples of C. Otherwise, A and B are *incommensurable*. Notice that Definition 2 only applies to "straight lines" (that is to say, only applies to lengths of line segments). Two lines segments with lengths A and B, respectively, are *commensurable in square* if A^2 and B^2 (which represent magnitudes of area) are commensurable in the sense of Definition 1, otherwise A and B are incommensurable. This has the implication (weird. . . at least to us) that $\sqrt{2}$ and $\sqrt{3}$ are commensurable in square, since $3 \times (\sqrt{2})^2 = 2 \times (\sqrt{3})^2$; in fact, 1 and $\sqrt{n}$ are commensurable in square for all natural numbers n. Notice that if A and B are the lengths of commensurable line segments, then A and B are also commensurable in square. Since Euclid deals with volumes, it seems odd that he does not similarly define "commensurable in cube." In Definition 3, Euclid's use of the terms "rational" and "irrational" differently than mathematicians both before and after him. The usual use of the words correspond to commensurable and incommensurable, respectively (see David Joyce's "Guide" to these four definitions on his online *Elements* website; accessed 7/27/2023). By "rational" as applied to *lengths* of line segments, Euclid means commensurable in square (which can be determined once a unit length segment is defined). But "irrational," Euclid means not rational in this sense. This holds throughout Book X and makes reading it tricky! To further complicate things, Definition 4 implies that for *areas*, Euclid uses the terms "rational" and "irrational" in the same way that we do. The first proposition in Book X sets the stage for the method of exhaustion. Proposition X.1 is not used in the rest of Book X, but it plays a role in Book XII.

Proposition X.1. Two unequal magnitudes being set out, if from the greater there is subtracted a magnitude greater than its half, and from that which is left a magnitude greater than its half, and if this process is repeated continually, then there will be left some magnitude less than the lesser magnitude set out. And the theorem can similarly be proven even if the parts subtracted are halves.

This is half of the method of exhaustion, and will be put to full use by Archimedes in his derivations of such formulae as the volume of a circle, cylinder, and sphere (see Section 6.2. Archimedes and my online PowerPoint presentation on Supplement. Archimedes: 2,000 Year Ahead of His Time). In fact, Proposition X.1 is an "epsilon property." The "lesser magnitude set out" corresponds to a $\varepsilon > 0$. The plan is to approximate "the greater" magnitude by recursively chipping off pieces of a magnitude at least half the size of what's left between the greater magnitude and the (recursively built up) lesser magnitude. Then summing up the magnitudes of the pieces we get a magnitude within ϵ of the greater magnitude (and less than the greater magnitude). For the other half of the method of exhaustion, we would start with the greater magnitude and similarly chip off pieces to get a greater magnitude within ϵ of the lesser magnitude (and greater than the lesser magnitude). For a given magnitude M, we first treat it as the greater magnitude and find magnitude m with $M - m < \varepsilon$, then treat M as the lesser magnitude and find magnitude M' such that $M' - M < \varepsilon$. In either case, we find a magnitude with ϵ of M. This is very-much in the spirit of of a limit from Calculus 1. Lemma 1 for Proposition X.29 states: "To find two square numbers such that their sum is also square." Three natural numbers x, y, z form a *Pythagorean triple* if $x^2 + y^2 = z^2$. The proof of the lemma shows that if m and n are natural numbers of the same parity (i.e., both are even or both are odd) with $m < n$, then $x = mn$, $y = (n^2 - m^2)2\text{m}$ and $z = (n^2 + m^2)/2$ form a Pythagorean triple. A *primitive Pythagorean triple* is one where x, y, and z share no common multiples. If m and n are relatively prime odd numbers, then the resulting x, y, z form a primitive Pythagorean triple. These ideas are discussed in Elementary Number Theory (MATH 3120); see my online notes for this class on Section 16. Pythagorean Triangles (notice Theorem 16.1, but the roles of m and n are slightly different there). We leave the rest of Book X for "the reader to explore(!)," as Eves does (Eves only devotes a paragraph to Book X). For those interested, a useful source companion in attempts to read Book X is D. H. Fowler's "An invitation to read Book X of Euclid's *Elements*," *Historia Mathematics*, **19**(3), 233-264 (1992), which can be viewed online on the the ScienceDirect website (accessed 7/27/2023).

Note 5.4.P. The last three books, XI, XII, and XIII, of the *Elements* cover solid geometry (that is, the geometry of three dimensional objects). These books "cover much of the material, with the exception of that on sphere, commonly found in high-school texts" (according to Eves on page 149). However, your humble instructor's high-school geometry class (taken during academic year 1978–79), these topics were restricted to a 20 page final chapter of the textbook; see my (tentative and partial) online notes on High School Geometry based on the 1970s book *Modern School Mathematics: Geometry.* Book XI covers lines and planes in space, and parallelepipeds. It contains twenty-eight definitions and thirty-nine propositions. Some of the definitions are (these are straightforward and don't need elaboration):

Definition 1. A solid is that which has length, breadth, and depth.

Definition 6. The inclination of a plane to a plane is the acute angle contained by the straight lines drawn at right angles to the intersection at the same point, one in each of the planes.

Definition 8. Parallel planes are those which do not meet.

Definition 14. When a semicircle with fixed diameter is carried round and restored again to the same position from which it began to be moved, the figure so comprehended is a sphere.

Definition 18. When a right triangle with one side of those about the right angle remains fixed is carried round and restored again to the same position from which it began to be moved, the figure so comprehended is a cone. And, if the straight line which remains fixed equals the remaining side about the right angle which is carried round, the cone will be right-angled; if less, obtuse-angled; and if greater, acute-angled.

Definition 28. A dodecahedron is a solid figure contained by twelve equal, equilateral and equiangular pentagons.

Some of the propositions are:

Proposition XI.2. If two straight lines cut one another, then they lie in one plane; and every triangle lies in one plane. [This is the familiar fact that three points determine a plane; as an application, this is why a tripod has three legs!]

Proposition XI.6. If two straight lines are at right angles to the same plane, then the straight lines are parallel. [This is interesting, since it is not true in spaces of dimension greater than three.]

Proposition XI.29. Parallelepipedal solids which are on the same base and of the same height, and in which the ends of their edges which stand up are on the same straight lines, equal one another. [This and the next given proposition are conditions implying parallelepipes have equal volumes.]

Proposition XI.31. Parallelepipedal solids which are on equal bases and of the same height equal one another.

Book XII is relatively brief with no new definitions and eighteen propositions. It mostly concerns volumes spheres, pyramids, cones, and cylinders. Several of the propositions are proved with the method of exhaustion, which we consider in more detail in Section 11.3. Eudoxus' Method of Exhaustion. Some results are:

Proposition XII.2. Circles are to one another as the squares on their diameters. [That is, the area of a circle is proportional to the square of the diameter, and hence also proportional to the square of the radius. We now denote the constant of proportionality between the area and the square of the radius as π.]

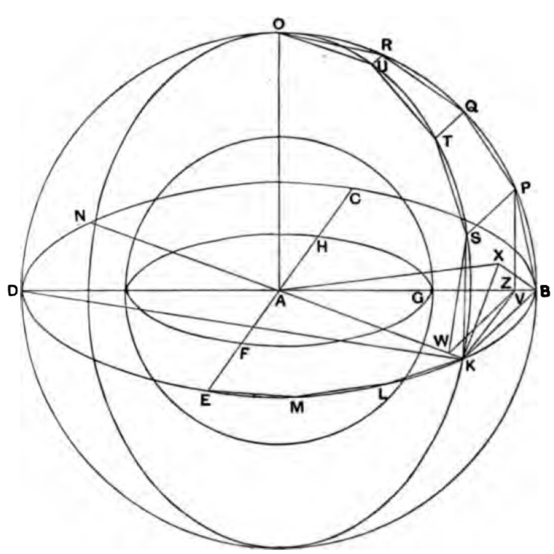
Proposition XII.6. Pyramids of the same height with polygonal bases are to one another as their bases. [That is, such pyramids have volumes proportional to the areas of their base.]

Proposition XII.10. Any cone is a third part of the cylinder with the same base and equal height. [With the volume of a cylinder as as $\pi r^2 h$, this implies that the volume of a cone is $\frac{1}{3}\pi r^2 h$.]

Proposition XII.16. Given two circles about the same center, to inscribe in the greater circle an equilateral polygon with an even number of sides which does not touch the lesser circle. [That is, for two concentric circles this gives a construction for inscribing a polygon with an even number of sides inside the larger circle which does not intersect the inner circle.]

Proposition XII.17. Given two spheres about the same center, to inscribe in the greater sphere a polyhedral solid which does not touch the lesser sphere at its surface. [This is similar to the previous proposition, though for spheres instead of circles and for polyhedra instead of polygons.]

The image used in the proof of Proposition XII.17 as given in Heath's translation of the *Elements* as follows (page 425):



Note 5.4.Q. The final book, Book XIII, contains 18 propositions and one "concluding remark." After several propositions on cutting straight line (segments), properties of equilateral and equiangular pentagons, and inscribing such pentagons in circles, Euclid presents the constructions of the five regular polyhedra (that is, the Platonic solids) mentioned in Section 3.9. The Regular Solids: the tetrahedron, cube, octahedron, dodecahedron, and icosahedron. The relevant propositions are:

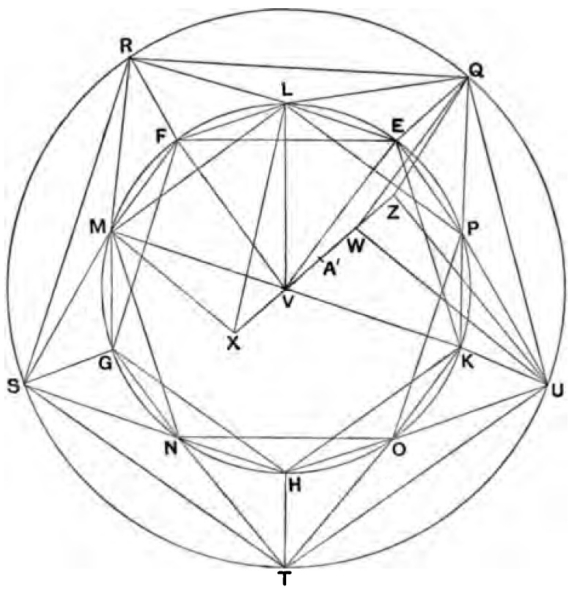
Proposition XIII.13. To construct a pyramid, to comprehend it in a given sphere; and to prove that the square on the diameter of the sphere is one and a half times the square on the side of the pyramid. [Construction of a tetrahedron.]

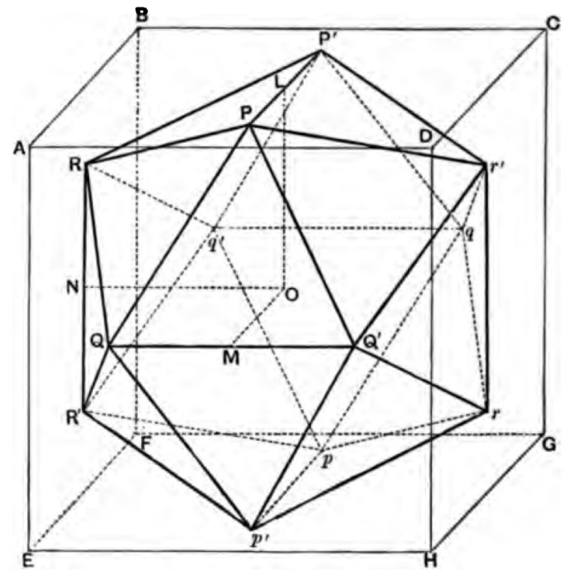
Proposition XIII.14. To construct an octahedron and comprehend it in a sphere, as in the preceding case; and to prove that the square on the diameter of the sphere is double the square on the side of the octahedron.

Proposition XIII.15. To construct a cube and comprehend it in a sphere, like the pyramid; and to prove that the square on the diameter of the sphere is triple the square on the side of the cube.

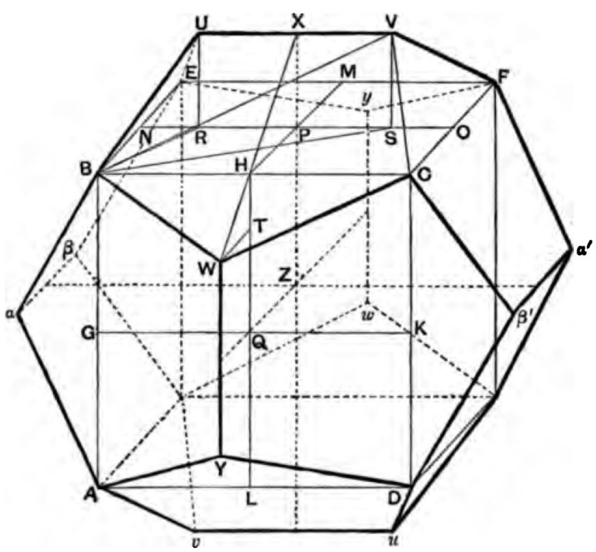
Proposition XIII.16. To construct an icosahedron and comprehend it in a sphere, like the aforesaid figures; and to prove that the square on the side of the icosahedron is the irrational straight line called minor. [See the figures below.]

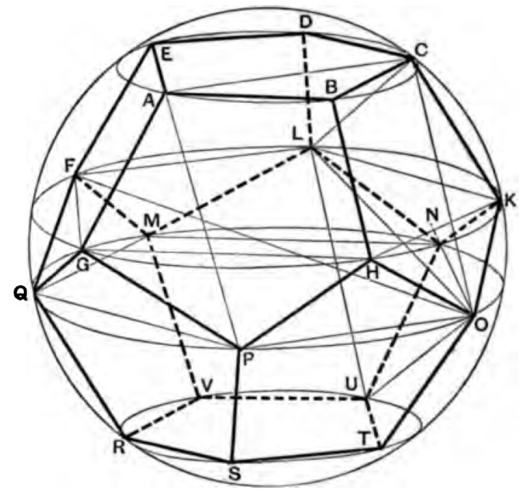
Proposition XIII.17. To construct a dodecahedron and comprehend it in a sphere, like the aforesaid figures; and to prove that the square on the side of the dodecahedron is the irrational straight line called apotome. [See the figures below.]





Images for Proposition XIII.16 from Heath's translation, pages 481 and 492





Images for Proposition XIII.17 from Heath's translation, pages 499 and 502

Concluding Remark. I say next that no other figure, besides the said five figures, can be constructed by equilateral and equiangular figures equal to one another. We need convexity to get the claimed uniqueness. Euclid never states this, so it is an unstated assumption. To see the problem, consider five of the adjacent triangles in the icosahedron. These can be "punched down into" the icosahedron to make an equilateral and equiangular figure with twenty triangular sides which is different from the icosahedron (called a "punched-in icosahedron; but is not convex). Robin Hartshorne offers other critiques, and solutions, to Euclid's existence and uniqueness claims of Book XIII. In his Chapter 8 ("Polyhedra"), Section 44 ("The Five Regular Solids") of his *Geometry: Euclid and Beyond* (Springer, 2000), he proved the following:

Theorem 44.4. Any polyhedron that is

(a) bounded by equal regular polygons,

(b) convex,

(c) has the same number of faces at each vertex,

is congruent (up to a scale factor) to one of the five: tetrahedron, cube, octahedron, icosahedron, dodecahedron. Furthermore, these five all have the additional properties:

(d) all dihedral angles are equal,

(e) the vertices lie on a sphere, and

(f) for any two vertices, there is a rigid motion of the figure taking one to the other.

Note. Several of the explanations of the meanings of Euclid's definitions and propositions are based on the "Guide" portion of David Joyce's online *Euclid's Elements* (accessed July 2023).

Note 5.4.R. The fact that the *Elements* conclude with the existence and uniqueness proofs for the regular solids has lead to speculation that the whole purpose of this work of Euclid's is to reach this conclusion. Eves states (page 149): "The frequently stated remark that Euclid's *Elements* was really intended to server merely as a drawn-out account of the five regular polyhedra appears to be a lopsided evaluation. More likely, it was written as a beginning text in general mathematics. Euclid also wrote texts on higher mathematics." Alexey Stakhov (assisted by Scott Olsen) in his *The Mathematics of Harmony: From Euclid to Contemporary Mathematics and Computer Science*, Series on Knots and Everything, Volume 22, World Scientific Publishing (2009), says on his page xxvii:

> "According to Proclus' opinion, Euclid created the *Elements* not with the purpose to present geometry as axiomatic mathematical science, but with the purpose to give the full systematized theory of Platonic Solids, in passing having covered some advanced achievements of the ancient mathematics. Thus, the main goal of the *Elements* was a description of the theory of Platonic Solids described in the final book of *Elements*."

In my search of *Proclus: A Commentary on the First Book of Euclid's Elements*, translated by Glenn R. Morrow (Princeton University Press, 1970), the only relevant comment about this I can find is on page 57 of this book: "Euclid belonged to the persuasion of Plato and was at home in this philosophy; and this is why he [Eratosthenes, I think] thought the goal of the Elements as a whole to be the construction of the so-called Platonic figures." This claim seems improbable, because there is so much material in the *Elements* that is not needed for Book XIII. Eves' claim that is is a "text in general mathematics" of the time seems much more probable. Evidence of the thoroughness of Euclid's *Elements* is given by the fact that it effectively replaced previous geometry works, giving us the near-vacuum of the history of Greek geometry before Euclid. In Supplement. Proclus Commentary on Eudemus' *History of Geometry* we cover the little that *is* known about Euclid's predecessors.

Revised: 4/30/2024