5.5. The Theory of Proportion

Note. In this section we consider the theory of proportions of (1) the Pythagoreans, (2) Eudoxus, and (3) a modern view. To compare these, we give three proofs of Proposition 1 of Book VI of the *Elements*.

Note. Recall that Proposition VI.1 is the basis for thirty-one of the other thirty-two proofs in Book VI (see Section 5.4. Content of the "Elements", Note 5.4.K). The statement is:

Proposition VI.1. Triangles and parallelograms which are under the same height are to one another as their bases.

The "are to one another as their bases" means that they have areas that are in the same proportion as the lengths of their bases. We take Proposition 38 of Book I as given, which states: "Triangles which are on equal bases and in the same parallels equal one another." That is, if two triangles have equal bases and equal heights (or "parallels") then they have equal areas. From this we can also conclude (by Common Notion 5, "the whole is greater than the part," if you like) that if two triangles have the same height, then the if one has a greater base then it also has a greater area. For the proofs, we let the triangles be $\triangle ABC$ and $\triangle ADE$, with bases BC and DE lying on the same line MN, as in the figures below.

Note 5.5.A. The Pythagoreans (before the discovery of irrationals/incommensurables) would assume that the line segments BC and DE are commensurable. Their "proof" would then proceed as follows.

"Proof." With BC and DE assumed to be commensurable, let p and q be positive integers such that p times the length of BC equals q times the length of DE (that is, $BC : DE = p : q$). Mark off these points of division and connect them to point A creating q triangles on line segment BC and p triangles on line segment DE, where each triangle has the same length base, namely $BC/q = DE/p$ (see Figure 38 below). By Proposition I.38, each of these triangles have the same area. Therefore $\triangle ABC : \triangle ADE = p : q = BC : DE$, as claimed. *Q.E.D.*

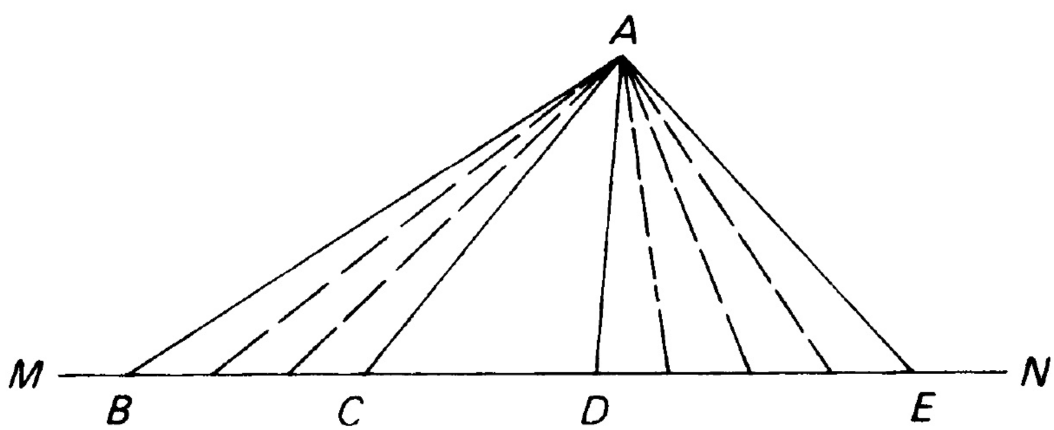

FIGURE 38

Note. Since the Pythagorean "proof" assumes that $BC/DE = p/q$ is rational (since the Pythagoreans denied the existence of irrational numbers, at least initially), then the proof is not valid in general. However, the proof given in Book VI uses Eudoxus' theory of proportion and avoids this assumption.

Note 5.5.B. The proof given in the *Elements* is as follows.

Proof. Let m and n be positive integers. On line MN, starting at point B, mark off successively $m-1$ segments equal to CB and connect the points of division, denoted $B_2, B_3, \ldots, B_m$, to vertex A, as shown in Figure 39 below). Similarly, on line MN, starting at point E, mark off successively $n-1$ segments equal to DE and connect the points of division, denoted $E_2, E_3, \ldots, E_n$, to vertex A. Then $B_mC = m(BC)$, $\triangle ABMC = m(\triangle ABD)$ (by Proposition I.8), $DE_n = n(DE)$, and $\triangle ADE_n = n(\triangle ADE)$ (also by Proposition I.38). By Proposition I.38 and Common Notion 5, if $B_mC \le DE_n$ (respectively $B_mC \ge DE_n$) if and only if $\triangle AB_mC \le \triangle ADE_n$ (respectively, $\triangle AB_mC \ge \triangle ADE_n$) and if and only if $m(\triangle ABC) \le n(\triangle ADE)$ (respectively, $m(\triangle ABC) \ge n(\triangle ADE)$).

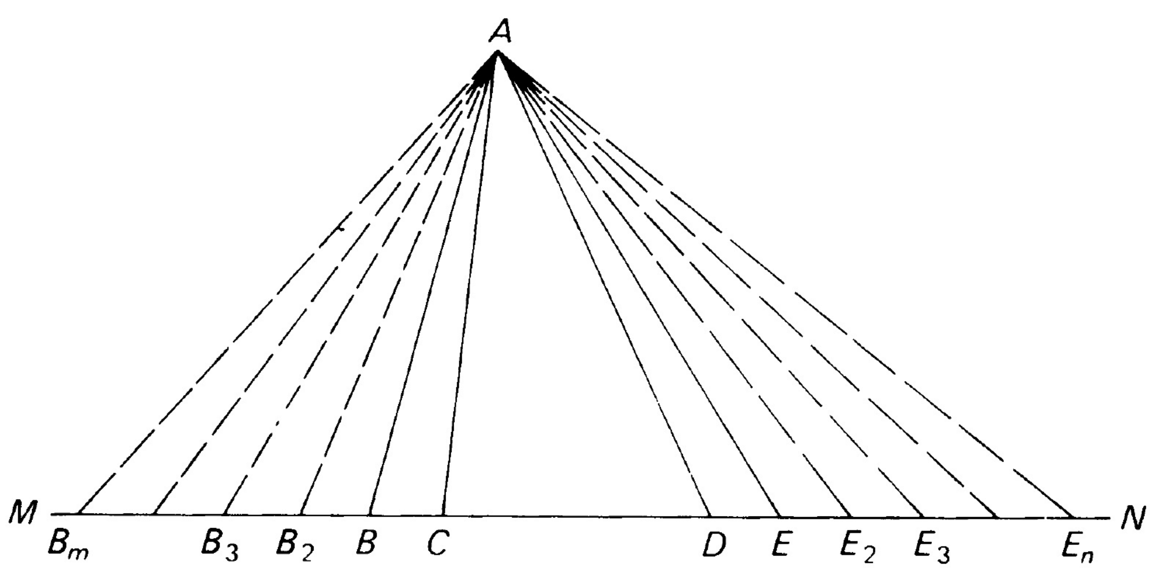

FIGURE 39

That is, $m(BC) \leq n(DE)$ (respectively, $m(BC) \geq n(DE)$) if and only if $m(\triangle ABC) \leq n(\triangle ADE)$ (respectively, $m(\triangle ABC) \geq n(\triangle ADE)$). Hene, by the Eudoxian definition of proportion (see Definition 5 of Book V and Section 5.4. Content of the "Elements", Note 5.4.J), $\triangle ABC : \triangle ADE = BC : DE$, as claimed. *Q.E.D.*

Note 5.5.C. We now give a proof based on modern analytic techniques. We assume the result holds when $BC/DE = p/q$ is rational, as shown in Note 5.5.A.

Proof. We only need now to consider the case when BC and DE are incommensurable; that is, when BC/DE is irrational. Let n be a positive number that is greater than BC/DE. Divide line segment BC into n equal parts, with BR being one of the parts, as shown in Figure 40 below. (Notice that $BR = BC/n < BC/(BC/DE) = DE$.) On line segment DE mark off successively segments of length equal to BR, arriving at a point F on DE such that $FE < BR$ (since $BR < DE$, this can be done).

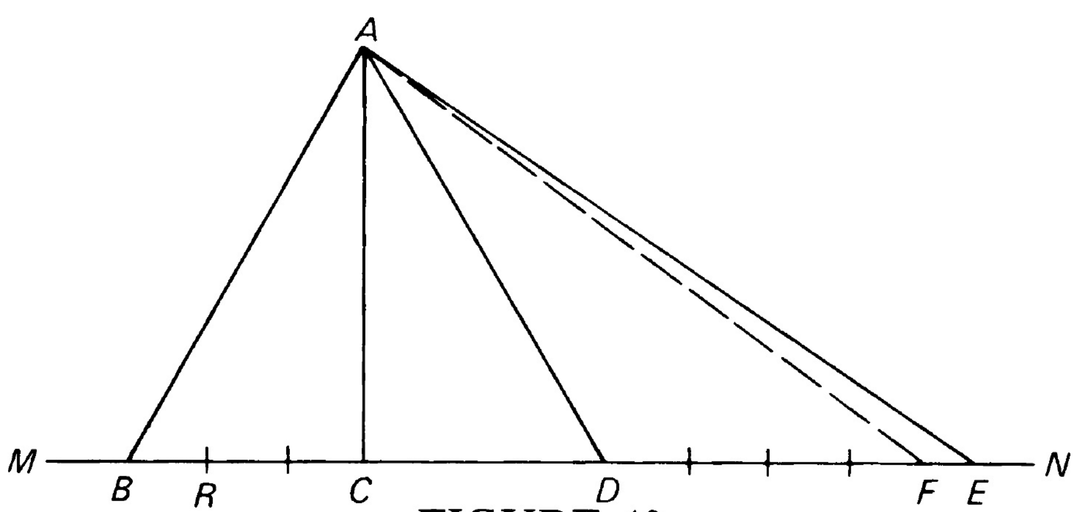

FIGURE 40

By the commensurable case (in Note 5.5.A), we have $\triangle ABC : \triangle ADF = BC : DF$. Since this holds for arbitrary n sufficiently large, then we take a limit as $n \to \infty$. Now for $n \to \infty$, we have $DF \to DE$ and $\triangle ADF \to \triangle ADE$. Therefore, $\lim_{n\to\infty}(\triangle ABC : \triangle ADF) = \lim_{n\to\infty}(BC : DF)$, or $\triangle ABC : \triangle ADE = BC : DE$, as claimed. *Q.E.D.*

Note. The proof given in Note 5.5.C depends on approximating the irrational BC/DF with a rational number. The approximation in terms of the number n in the proof is n/m where m is the number of segments of length equal to BR that are used in marking off points on line segment DE to find point FE. Since the length of BR is BC/n, then $m = \lfloor DE/(BC/n)\rfloor = \lfloor n(DE/BC)\rfloor$ (the symbols here mean to round down) and $mn \approx DE/BC$ where the approximation is better when n (and then, necessarily m also) are "big."

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