5.6. Regular Polygons

Note. In this section we consider (again) the constructions of regular polygons given in Euclid's *Elements*. We also give necessary and sufficient conditions for the construction of a regular n-gon with a compass and straight edge.

Note. We saw in Section 5.4. Content of the "Elements" that Euclid addressed constructions of regular n-gons for $n \in \{3,4,5,6,15\}$ (see Note 5.4.I). Since an angle can be bisected by Proposition I.9, if an n-gon can be construction with a compass and straight edge, then a $2^m n$-gon can be constructed by repeated bisection (m times per angle). So it follows from the *Elements* that for all integers $n \geq 0$, a regular polygon can be constructed with the number of sides as 2^n (here we need $n \geq 2$), $3(2^n)$, $5(2^n)$, or $15(2^n)$. This remained the list of constructible polygons until 1796 when Carl Friedrich Gauss (April 30, 1777–February 23, 1855) showed that a regular 17-gon could be constructed with a compass and straight edge. As opposed to giving a direct construction, he showed that a 17-gon can be constructed from a line segment of length

$$\frac{1}{16}\left[-1+\sqrt{17}+\sqrt{34-2\sqrt{17}}+\sqrt{68+12\sqrt{17}-16\sqrt{34+2\sqrt{17}}-2(1-\sqrt{17})\sqrt{34-2\sqrt{17}}}\right].$$

We have already referenced Section 14.2. Impossibility of Solving the Three Famous Problems with Euclidean Tools, in which constructible numbers are classified (see Theorem 32.6 in that section). With this classification, we see that Gauss's line segment is constructible. He published this result in Section 17 of his *Disquisitiones Arithmeticae* ("Investigations in Arithmetic") in 1801. Herbert W. Richmond (July

17, 1863–April 22, 1948) found a simple direct construction in 1893. It seems surprising that a question addressed in Euclid's *Elements* sat dormant until the 19th century! Gauss went further and gave conditions for the construction of certian regular polygons. However, he did not show the conditions were necessary. The problem was completely solved by Pierre Wantzel (June 5, 1814–May 21, 1848) in 1837 and published as "Recherches sur les moyens de reconnâitre si un Problème de Géométrie peut se réesoudre avec la règle et le compas" ("Research on Ways to Recognize if a Problem of Geometry can be Solved with Ruler and Compass") in *Journal de Mathèmatiques Pures et Appliquées* **1**(2), 366–372. In this paper he also proved the impossibility of doubling the cube and trisecting and angle. More on this and a biography of Wantzel is given in the previously mentioned Section 14.2. Impossibility of Solving the Three Famous Problems with Euclidean Tools.

Note. To discuss necessary and sufficient conditions for the compass and straight edge construction of a regular n-gon requires a new definition.

Definition. A prime number of the form $2^{(2^k)} + 1$ for non-negative integer k is a *Fermat prime.*

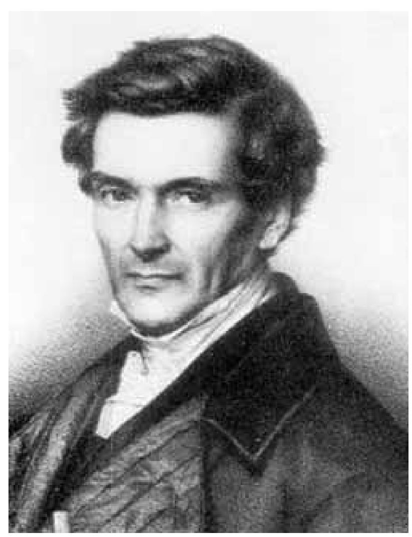
Note. The only known Fermat primes are 3, 5, 17, 257, and 65,537 which correspond to $k = 0, 1, 2, 3, 4$, respectively. For $5 \leq k \leq 19$, $2^{(2^k)} + 1$ is a composite number. It is unknown if $k = 20$ produces a composite or a prime number. It is unknown whether the number of Fermat primes is finite or infinite. More details on Fermat primes can be found in my online notes for Mathematical Reasoning (MATH 3000) on Section 6.7. More on Prime Numbers.

Note. The necessary conditions of Wantzel and the sufficient conditions of Gauss combine to give the following:

Theorem. Gauss and Wantzel. The regular n-gon is constructible with a compass and straight edge if and only if all the odd primes dividing n are Fermat primes whose squares do not divide n. This condition on n implies that $n = 2^k p_1 p_2 \cdots p_t$ where each p_i is a distinct Fermat prime.

This classifies the constructible regular polygons, with the exception that the collection of Fermat primes remains unknown!

Note. The theorem of Gauss and Wantzel appears in Introduction to Modern Algebra 2 (MATH 4137/5137) as Theorem 55.8 of Section X.55. Cyclotomic Extensions. The equipment needed to prove this includes field theory, algebraic extension fields, Galois theory, cyclotomic polynomials, and cyclotomic extensions. This is standard material (time permitting) in a second senior-level abstract algebra class. The following images are from the MacTutor biography webpages on Carl Friedrich Gauss (left) and Pierre Wantzel (right); accessed 7/29/2023.



Note. The first explicit construction of a regular 257-gon was given in 1822 by Magnus Georg Paucker (November 26, 1787–August 21, 1855) in "Geometrische Verzeichnung des regelmäßigen Siebzehn-Ecks und Zweyhundersiebenundfünfzig-Ecks in den Kreis" ("Geometric construction of the regular seventeen-corner and two hundred and fifty-seven corners in the circle"), *Jahresverhandlungen der Kurländischen Gesellschaft für Literatur und Kunst* ("*Annual proceedings of the Courland Society for Literature and Art*"), **2**, 160–219 (1822). Antoerh construction was given in 1832 by Friedrich Julius Richelot (November 6, 1808–March 31, 1875) in Friedrich Julius Richelot (1832). "De resolutione algebraica aequationis $x^{257} = 1$, sive de divisione circuli per bisectionem anguli septies repetitam in partes 257 inter se aequales commentatio coronata" ("On the algebraic resolution of the equation $x^{257} = 1$, or on the division of a circle by the bisection of an angle repeated seven times into 257 equal parts", in Latin), *Journal fü die reine und angewandte Mathematik* (*Journal for Pure and Applied Mathematics*), **9**: 1–26, 146–161, 209–230, 337–358 (1832). A construction for a regular 65,537-gon was first given by Johann Gustav Hermes (June 20, 1846–June 8, 1912) in 1894. He spent 10 years completing the 200-page manuscript. He published an announcement of his work in "Über die Teilung des Kreises in 65537 gleiche Teile" ("On dividing the circle into 65537 equal parts"), *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse* (*News from the Society of Sciences in Göttingen, mathematical-physical class*), **3**: 170–186 (1894). The information in this note is from the Wikipedia webpage on Constructible Polygons (accessed 7/29/2023).

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