5.7. Formal Aspect of the "Elements"

Note. In this section we consider axiomatic systems, undefined terms, and axioms/postulates in light of Euclid's use of them in the *Elements*. Eves is very brief about this, and this section of notes includes many links to notes for related classes.

Note. Eves states on page 152: "...Euclid's *Elements* has become the prototype of modern mathematical form. Certainly one of the greatest achievements of the early Greek mathematicians was the creation of the postulational form of thinking." His definition/postulate/theorem/proof approach is followed in pretty much every pure math text today. This is most prominently the case in ETSU's Analysis 1 (MATH 4217/5217), Introduction to Modern Algebra (MATH 4127/5127), Introduction to Topology (MATH 4357/5357), the set theory component of Mathematical Reasoning (MATH 3000; see Chapter 2), and of course Introduction to Modern Geometry (MATH 4157/5157). Another class that would take this approach is Introduction to Set Theory (though ETSU does not have this as an official class).

Note. As you see in Introduction to Modern Geometry (MATH 4157/5157) a necessary part of an axiomatic system is a set of undefined terms (see Note 1.3.A in my online notes for that class on Section 1.3. Axiomatic Systems. Since things can only be defined in terms of other things, then at some point we must have some fundamental terms (or objects) that remain undefined. Without undefined terms, we would ultimately end up with useless, circular definitions. We see the futility of trying to define everything in some of Euclid's definitions in Book I (Definitions 1 and 2): "A *point* is that which has no part. A *line* is breadthless length." "Part"? "Breadthless length"? What the hell does that mean? So we are stuck with fundamental, undefined terms. These terms are given there meaning by the axioms of the axiomatic system and by the logical implications of those axioms (i.e., the theorems/propositions). A point is not a dot on a piece of paper or a white board, nor is a line something "straight" drawn on a writing surface. These are *ideas* that only have the meaning given in the axiomatic systems (and the images you hold in your mind of tiny dots and fine, straight etchings are representations of a *model* of Euclidean geometry). **This is how mathematics works!!!**

Note. Greek mathematicians (including Euclid) made a distinction between a "postulate" and an "axiom." This is not how modern math uses the terms, and today they are interchangeable. Eves describes the distinctions as follows (quoting form page 153):

1. An axiom is a self-evident assumed statement about something, and a postulate is a self-evident assumed construction of something; thus axioms an postulates bear a relation to one another much like that which exists between theorems and construction problems.

2. The axiom is an assumption common to all sciences, whereas a postulate is an assumption peculiar to the particular science being studied.

3. An axiom is an assumption of something that is both obvious and acceptable to the learner; a postulate is an assumption of something that is neither obvious nor necessarily acceptable to the learner. ...In modern mathematics, no distinction is made, nor is the quality of being self-evident or obvious considered. ...

Eves distinguishes between axioms and postulates as set out in the *Elements* by referring to the five "Common Notions" as five axioms, and then then following the original terminology used by Euclid for postulates.

Note. Just as we needed undefined foundational terms, we also need some unproved foundational assumptions. These assumptions are called (in modern times) *axioms* or *postulates*. These can be motivated by the real physical world, or not. For example, Playfair's Axiom ("Given a line an a point not on the line, it is possible to draw exactly one line through the given point parallel to the line"; see Note 5.3.I in Section 5.3. Euclid's "Elements") is intuitive and the how "straight lines" on a piece of paper behave. However the negation of Playfair's Axiom, "Given a line an a point not on the line, it is possible to draw more than one line through the given point parallel to the line," is unintuitive and arguably not inspired by the "real world" (you might even think that it is false...it *is false* in Euclidean geometry). However, there is an axiomatic structure, namely hyperbolic geometry, in which this holds true. This means that lines and points interact differently in hyperbolic geometry than they do in Euclidean geometry. It is not that one is right and the other wrong. They are both perfectly valid kinds of geometry. The concerns over an axiomatic system are not related to it being "right or wrong," but over it being *consistent* (i.e., it does not lead to contradictions), *independent* (i.e., there are redundant axioms), and *complete* (i.e., all meaningful statements within the system can be proved to be either true or false). These concepts are addressed in Introduction to Modern Geometry (MATH 4157/5157) in Section 1.3. Axiomatic Systems.

Note. Euclid's *Elements* is not without flaws (by modern standards, at least). There is no clear idea that the Euclidean plane forms a continuum, for example. As we saw Section 4.4. The Euclidean Tools, if we restrict ourselves to compass and straight edge motivated constructions, then we need not have a continuum (see Note 4.4.B)! The easiest way to resolve this is to associate real numbers with the points on a line, as René Descartes (March 31, 1596–February 11, 1650) will do in his analytic geometry in the 17th century (see Section 10.2. Descartes). Of course this requires a clear definition of what the real numbers are (they are a complete ordered field) and this idea is not solidified until the mid 19th century. Another concern is that of "betweenness." This too can be dealt with using the real numbers and the ordering of them (given by $<$ and $>$). For a convincing word of warning, see the "proof" of the obviously absurd "theorem" that states every triangle is isosceles. This and the related ideas of this note are address in Introduction to Modern Geometry (MATH 4157/5157) in Section 2.2. A Brief Critique of Euclid.

Note. Eves introduces two new terms on page 154: "The development [in the *Elements*] is the *synthetic* one of proceeding from the known and simpler to the unknown and more complex. Without a doubt, the reverse process, called *analysis*, of reducing the unknown and more complex to the known, played a part in the discovery of the proofs of many of the theorems, but it plays no part in the exposition of the subject." This last observation is very common in math research publications (and probably *too* common in math text books) today. Clean, pristine proofs are presented with little evidence as to the motivation or insights that lead to the given proof. In concluding this section, we give a little recent history concerning the teaching of geometry at ETSU. The terms "synthetic" and "analytic" (in the sense of "analysis" given here) appeared in the ETSU catalogue description of the graduate-level class, Axiomatic and Transformational Geometry (MATH 5330). The catalog description in the 2014-15 ETSU Graduate Catalog was: "Axiomatic and finite geometries. Euclidean geometry (synthetic/analytic), transformational geometries, non-Euclidean and projective geometries." Unfortunately, the course was removed from the graduate catalog in 2015. The senior/graduate level course Introduction to Modern Geometry (MATH 4157/5157) was previously titled "Vector Geometry" and the description in the 1988–90 ETSU Graduate Catalogue was: "Projective geometry, affine geometry and affine transformation, Euclidean geometry, non-Euclidean geometries." It seems that "Vector Geometry" was split into Introduction to Modern Geometry (MATH 4157/5157) and Axiomatic and Transformational Geometry (MATH 5330) sometime in the 1990s. This makes Introduction to Modern Geometry (MATH 4157/5157) all the more important, since it is now ETSU's only class dealing with classical Euclidean geometry (but in a "modern" way). The only other upper-level geometry class at ETSU is graduate-level Differential Geometry (MATH 5310). This has been taught sporadically since the late 1990s by your humble instructor. I have online notes for this as "Differential Geometry (and Relativity)". These notes have as much of an emphasis on introduction to special and general relatively as they do on relativity (giving a nontraditional approach to differential geometry). A more traditional approach is given in my online notes for Differential Geometry (though these notes are a work in progress, as of summer 2023).

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