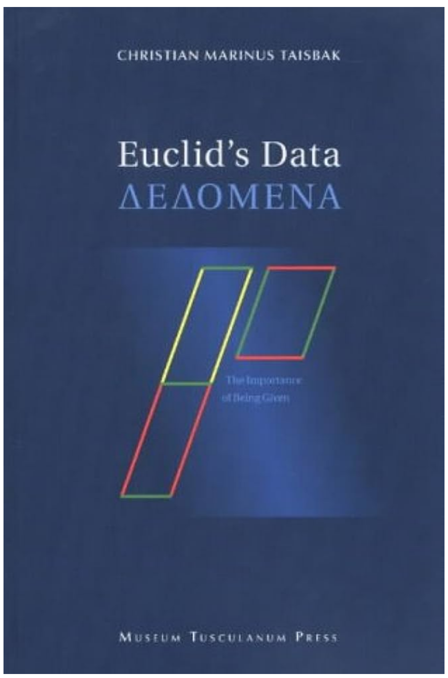
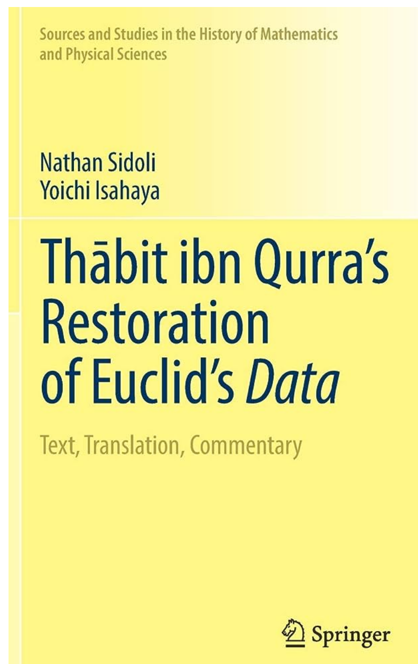
5.8. Euclid's Other Works

Note. In Section 5.2. Euclid we listed seven books other than the *Elements* which were authored by Euclid. In this section we give a few details on each and, when available, give sources for English translations of those that have survived. The sources for this section are Chapter II, Euclid's Other Works, of Thomas Heath's *The Thirteen Books of Euclid's Elements* (Cambridge University Press, 1925), and Thomas Heath's *A History of Greek Mathematics*, Volume I, From Thales to Euclid (Clarendon Press, 1921), Chapter XI, "Euclid."

Note 5.8.A. Euclid's *Data* concerns elementary geometry and may be regarded as elementary exercises in analysis. The form of the results are not to actually determine a thing or relation which is to be "given," but instead proves that it can be determined whether or not the thing or relation is given, once the hypotheses are known to hold (i.e., are "given"). So the propositions of the the *Data* could be stated either as theorems or problems. Eves describes this as (see page 154): "A *datum* may be defined as a set of parts or relations of a figure such that if all but any one are give, then that remaining one is determined." Much of the material of the *Elements* in Books I through VI is contained in the *Data*. The work was important enough that Pappus (circa 290 CE–350 CE) included a description of it in his *Treasury of Analysis*. This work survives in Greek and a translation from Arabic. An English translation of the Greek version is still in print by Christian Taisbak, *Euclid's Data: The Importance of Being Given* (Museum Tusculanum, 2003). Another English translation is Nathan Sidoli and Yoichi Isahaya's *Thābit ibn*

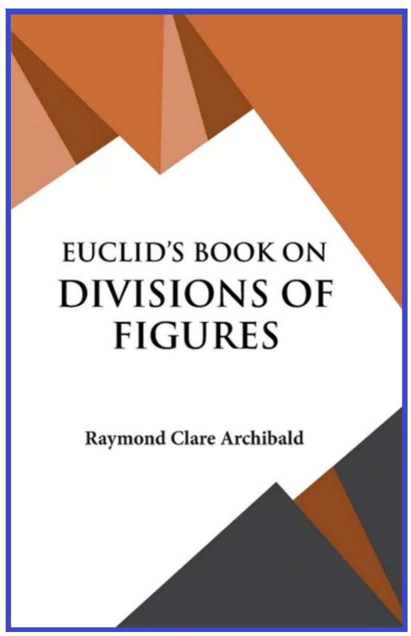
Qurra's Restoration of Euclid's Data, Text, Translation, Commentary (Sources and Studies in the History of Mathematics and Physical Sciences; Springer, 2018). This book discusses the differences between the Greek and Arabic versions, including a discussion of the concept of "given," as mentioned above. Both Thābit ibn Qurra's (836–February 18, 901) Arabic version and an English version are given; we met Thābit ibn Qurra in Section 5.3. Euclid's "Elements" in connection with this and other works of Euclid; see Note 5.3.F(4).





Note 5.8.B. Euclid's *On Divisions of Figures* is lost in Greek, but survives through Arabic translations. It primarily addresses construction problems which require the division of a figure by a restricted line (the restriction being passing through a given point, for example) so that the part will have areas in a prescribed ratio. Examples of such problems are given for quadrilaterals and trapezoids in Problem Study 3.11(b) and 3.11(c). Notice that a triangle can be divided by a line, either

into two triangles (called a division into "like figures") or into a triangle and a quadrilateral (called a division into "unlike figures"). Along with the *Elements* and the *Data*, this work is the only other known work of Euclid in pure geometry. Proclus (circa 411–April 17, 485) mentions *On Divisions* in his commentary on Book I of the *Elements*. An English version is Raymond Archibald's *Euclid's Book on Divisions of Figures*, with a Restoration Based on Woepcke's Text and on the *Practica Geometriae* of Leonardo Pisano (Cambridge University Press, 1915). This can be found online in PDF on Pavel Šišma's webpage at Masaryk University in the Czech Republic. This version is also still in print through Hawk Press (both websites accessed 7/30/2023).

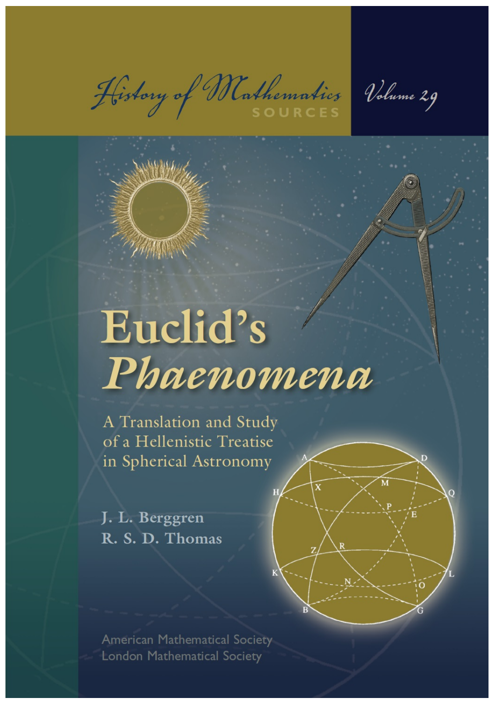


Heath gives the following history of the copies which we have of *On Divisions* in Volume 1 of his translation of the *Elements* (pages 8 and 9). First, English scholar John Dee (July 13, 1527–March 26, 1609) discovered a treatise *De divisionibus* by Muhammad Bagdadinus (or "Machometus Bagdadinus," died 1141). He passed a copy of it (in Latin) on to Italian mathematician Frederico Commandino (1506–

September 5, 1575) in 1563. Commandino published the work in Dee's name and his own in 1570. There is some uncertainty as to the origin of the Latin translation of the Arabic version. It is probably due to Gherard of Cremona (1114–1187), since the list of his translations include a "*liber divisionum*." A detailed version of the is in Paul Rose's "Commandino, John Dee, and the *De Superficierum Divisionibus* of Machometus Bagdedinus," *Isis*, **63**(1), 88–93 (1972). The Arabic version contains mistakes and unmathematical expressions, and does not contain a proposition about the division of a circle that Proclus mentions, so the Arabic version cannot be a direct translation of Euclid's original and does not contain all of the original material. German historian and mathematician Franz Woepcke (May 6, 1826–March 25, 1864) found an Arabic manuscript in Paris on the division of figures, which he translated into French and published in 1851. The manuscript attributes the work to Euclid and ti corresponds to the description given by Proclus. Only four of the thirty-six propositions of the Arabic manuscript include proofs (apparently the Arabic translator thought the other proofs were too easy and so did not include them). Otherwise, the work seems complete. Notice that Archilbald's English translation mentioned above is partially based on Woepcke's version (which includes proofs of all thirty-six propositions).

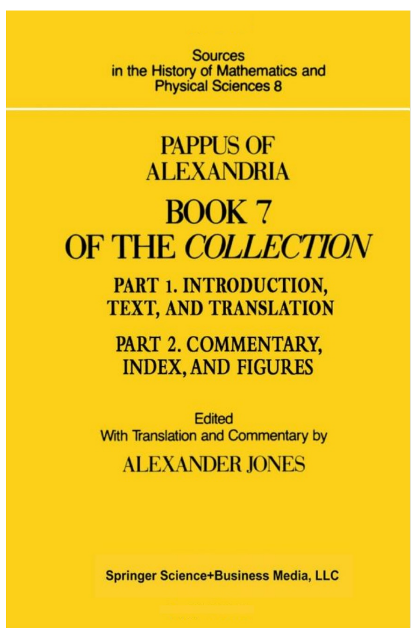
Note 5.8.C. The *Phaenomena* deals with the spherical geometry needed for the study of (ancient) astronomy. It consists of 18 propositions. Euclid seems to have based his work on *On the Moving Sphere* of Autolycus of Pitane (circa 360 BCE–circa 290 BCE) and possibly on a book of Eudoxus (403 BCE– 355 BCE) which contained only mathematical content (as opposed to astronomical applications).

Several sources exist containing parts of the work in *Phaenomena*, including codex (manuscript book) Vaticanus graecu 204. As a result, the *Phaenomena* still exists. It is in print today as J. Berggren and R. Thomas's *Euclid's Phaenomena*, A Translation and Study of a Hellenistic Treatise in Spherical Astronomy, History of Mathematics Sources, Volume 29 (American Mathematical Society/London Mathematical Society, 1996).



Note 5.8.D. Another work of Euclid's that is not concerned exclusively with math is his *Optics*. This contains results which today would be in the area of projective geometry. An English version of the *Optics* by Harry Burton as "The Optics of Euclid," *Journal of the Optical Society of America*, **35**(5), 357–72 (1945). This can be viewed online on Andrei Rodin's blog about History and Philosophy of Mathematics (accessed 7/30/2023). Unfortunately, Burton's paper gives no references and it is unclear what sources he used in his translation.

Note 5.8.E. The *Porisms* contained 38 lemmas and 171 theorems (or "porisms"), according to Pappus (circa 290 CE–circa 350 CE) in this *Treasury of Analysis* in Book VII of his *Collection*. In fact, the only original source of information on the *Porisms* is this work of Pappus (otherwise, the *Porisms* is lost). Eves defines a porism in a footnote on page 154 as follows: "A *porism* is taken today to be a proposition stating a condition that renders a certain problem solvable, and then the problem has infinitely many solutions. ... We do not know precisely Euclid's meaning of the term." Pappus gives two definitions in his *Collection*, and Proclus (circa 411–April 17, 485) in his *Commentary on Book I of the Elements* gives two meanings of *porism*. Pappus states (as quoted in Heath's translation of Euclid, Volume 1, page 10): "Now all the varieties of porisms belong, neither to theorems nor problems, but to a species occupying a sort of intermediate position..., the result being that, of the great number of geometers, some regard them as of the class of theorems, and others of problems, looking only to the form of the proposition."



In an attempt to clarify, we consider an example of a porism from *Pappus of*

Alexandria, Book 7 of the Collection, edited with translation and commentary by Alexander Jones (Springer, 1986). Jones states that this porism "apparently preserves Euclid's own wording" (see page 549 of Jones' book):

> "If lines from two given points inflect on a line given in position, and on (line) cuts off (an abscissa) from a line given in position up to a point given on it, the other (line) too will cut off from another (line given in position) and (abscissa) having a given ration (to the first)."

We now give Jones' interpretation of this porism. Consider two given lines h and e, point H on h, two points A and B on neither line, and a ratio $\alpha : \beta$ (in the figure below, left) Then it is possible to construct a line m (in red, below right) and a point M on m such that if variable lines a through A and b through B (both in blue, below right) intersect each other in a point E that lies on e, and the intersection of a with h is point G and the intersection of b with m is point N where the ratio of intervals $HG : MN$ equals $\alpha : \beta$ (HG and MN are in orange, below right).

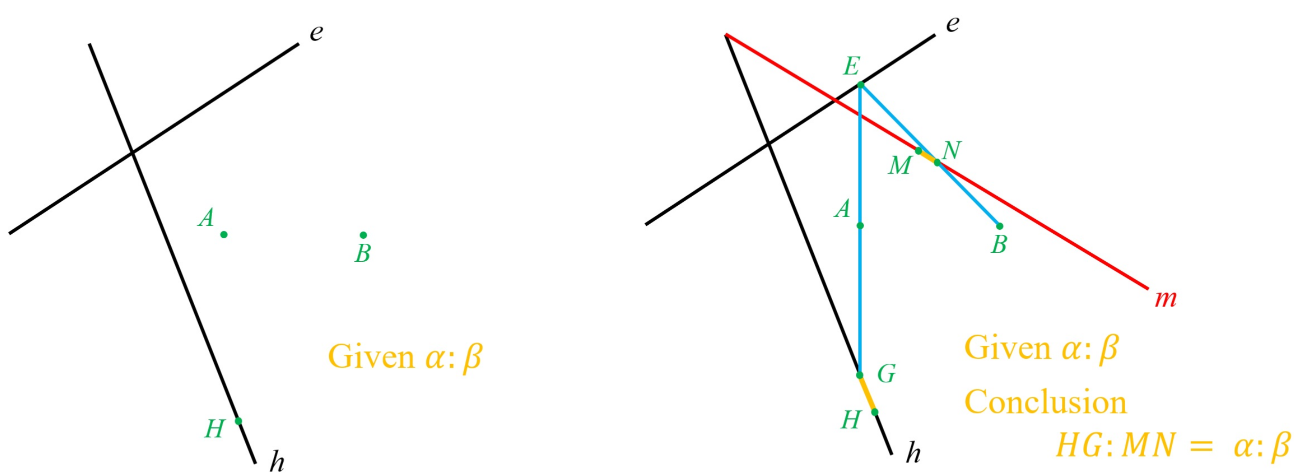


Figure. Based on Figure B.1 of Jones' *Pappus of Alexandria, Book 7 of the Collection*

Note 5.8.F. There are other lost books of Euclid. Their existence is known through commentaries of others. The *Pseudaria* is known through Proclus' (circa 411–April 17, 485) *Commentary on Book I of the Elements* in which he states (quoting from Heath's translation of the *Elements*, Volume 1, page 7): "[We have in this work] methods we shall be able to give beginners in this study practice in discovery of paralogisms, and to avoid being misled. . . . exercising our intelligence in each case by theorems of all sorts, setting the true side be side with the false, and combining the refutation of error with practical illustration." So it seems that the *Pseudaria* covered elementary geometry in an introductory (and possibly intuitive) way. The *Surface-loci* consisted of two books, which are mentioned by Pappus in his *Treasure of Analysis* in Book VII of his *Collection*. He states two lemmas from the treatise that lead us to conclude that it contained a consideration of cones, cylinders, and spheres. "Beyond this all is conjecture" (Heath, page 15). French geometer Michel Chasles (November 15, 1793–December 18, 1880) has conjectured that *Surface-loci* dealt with surfaces of revolution of the second degree and sections of them (so it would include ellipsoids and hyperboloids as well), as Heath states (on his page 16). Euclid's lost work *Conics* is known from Pappus' *Collection, Book VII* in which he states (quoting Heath, page 16): "The four books of Euclid's Conics were completed by Apollonius (circa 262 BCE–circa 190 BCE), who added four more and gave us the eight books of Conics." According to Pappus, Euclid's *Conics* is an improvement of an earlier work by Aristaeus the Elder ((circa 370 BCE–circa 300 BCE). Euclid's work then evidently was superseded by the work of Apollonius (we explore Apollonius and his *Conics* in Section 6.4. Apollonius). Finally, Euclid is mentioned in other manuscripts as having written *Elements of*

Music. Notice that such a work is not off topic as one might initially expect. Recall that the Pythagoreans had a theory of music and studied the numerical properties of musical intervals (see Section 3.2. Pythagoras and the Pythagoreans).

Revised: 7/31/2023