6.3. Eratosthenes

Note. In this section, we consider the work of Eratosthenes, who is best known for the "Sieve of Eratosthenes" and for determining the diameter of the Earth. The references for this section are the MacTutor biography webpage for Eratosthenes (from which we have the following image) and Thomas Heath's *A History of Greek Mathematics*, Volume 2 (Oxford: Clarendon Press, 1921), pages 104 to 109.

Image from the MacTutor biography webpage for Eratosthenes

Note. Eratosthenes (276 BCE–194 BCE) of Cyrene (modern day Shahhat, Libya) was well-versed in all branches of knowledge of his time. In addition to mathematics, he was distinguished as an astronomer, geographer, historian, philosopher, and poet (Eves, page 170). In 245 BCE, he went to Alexandria to tutor the son of Ptolemy III Euergetes and, in 240 BCE he became the head librarian of the Library of Alexandria. It is thought that he became blind in old age and died by suicide through voluntary starvation (Eves, page 170).

Note. Eratosthenes is known to have written *Platonicus*, which dealt with the mathematics which underlie Plato's philosophy. This work is now lost, but we know it from references by Theon of Smyrna (circa 70 CE–circa 135 CE). Twice Theon references *Platonicus* by name in his work on the mathematical background needed for students of Plato. Theon states that Eratosthenes studied the fundamental definitions of geometry and arithmetic, as well as the principles of music. Theon credits *Platonicus* as the source of the story of a king who wanted a cubical alter doubled in size (more details on this story are given in Section 4.5. Duplication of the Cube), leading to the doubling-of-the-cube construction problem.

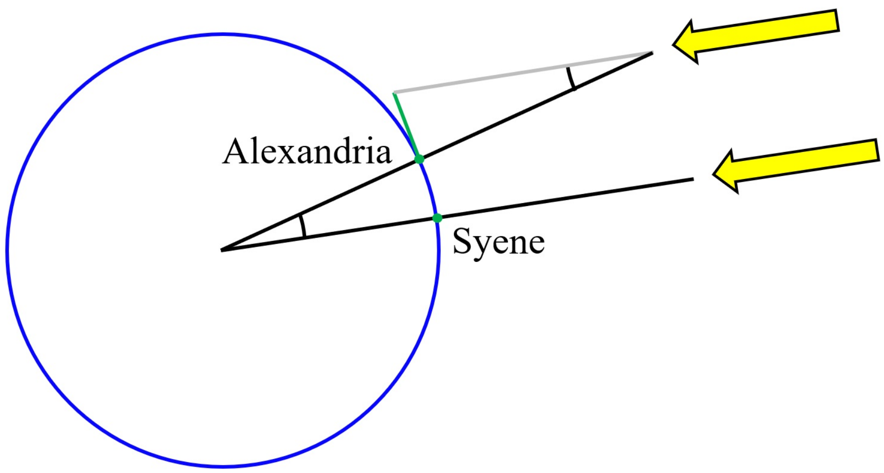
Note. Eratosthenes also wrote a work *On means*. This was in two books and important enough that Pappus (circa 290 CE–circa 350 CE) listed it along with works by Euclid, Aristaeus, and Apollonius as forming part of the *Treasury of Analysis* (evidence that *On means* was a systematic geometrical treatise). Pappus also mentions loci of points (that is, a curve made up of points) which Eratosthenes called "loci with reference to means," though it is uncertain as to what this meant.

Note. We know the "Sieve of Eratosthenes" from Nicomachus of Gerasa's (circa 60 CE–circa 120 CE) *Introduction to Arithmetic* (we will mention Nichomachus again in passing in Section 8.1. "The Dark Ages" (The Middle Ages); see Note 8.1.B). You may see it Elementary Number Theory (MATH 3120); see my online notes on Section 2. Unique Factorization. The idea the sieve of Eratosthenes is to find prime numbers by listing all natural numbers (up to n, say) and eliminating composite numbers by first removing multiples of 2 greater than 2, then removing multiples of 3 greater than 3, then removing multiples of 5 greater than 5, etc. If all multiples of primes less than or equal to $\sqrt{n}$ have been removed, then all numbers less than n which remain must be prime. This follows from Lemma 2.4 of the Elementary Number Theory notes on Section 2. Unique Factorization and it states: "If n is composite, then it has a prime divisor d such that $1 < d \leq \sqrt{n}$." Wikipedia has a nice animated GIF of the Sieve of Eratosthenes which finds all prime numbers between 1 and $n = 121$ by eliminating multiples of 2, 3, 5, 7, and $\sqrt{n} = 11$.

Note. Eratosthenes is best known for his accurate measurement of the circumference of the Earth. This work appeared in Earatosthenes' *On the Measurement of the Earth* (now lost). Eves describes this in Problem Study 6.1(c) as follows:

> **Problem Study 6.1(c).** Eratosthenes, in 240 B.C., made a famous measurement of the earth. He observed at Syene [modern day Aswan, Egypt], at noon at the summer solstice, that a vertical stick had no shadow, while at Alexandria (which he believed to be on the same meridian with Syene) the sun's rays were inclined $\frac{1}{50}$ of a complete circle to the vertical. He then calculated the circumference of the earth from the known distance of 5000 stades between Alexandria and Syene. Obtain Eratosthenes' result of 250,000 stades for the circumference of the earth. There is reason to suppose that an Eratosthenian stade was equal to about 559 feet. Assuming this, calculate from the above result the polar diameter of the earth in miles. (The actual polar diameter of the earth to the nearest mile is 7900 miles.)

The fact that a vertical stick casts no shadow in Syene at noon on the summer solstice indicates that the sun is directly overhead (with an altitude of 90°) at this point in time. If Alexandria and Syene are on the same meridian (that is, have the same longitude), then it is noon at the same time in both places. So by measuring the altitude of the sun at this same point in time allows us to use the curvature of the Earth and these two angles as given in the following figure:



Notice that the sun is so far way that we can approximate the rays of light from the sun as parallel. The indicated angles represent "$\frac{1}{50}$ of a complete circle." Eratosthenes gives a circumference around the poles as 26,468 miles. The actual value is 24,860 miles, so Eratosthenes has a percentage error of −1.6%. For an alternative presentation of this problem, see my online notes for introductory Astronomy ("PS 215" at Auburn University) on Chapter 3. Early Astronomy. Eratosthenes also estimated the distance to the moon as 780,000 stades and the distance to the sun as 804,000,000 stades. These estimates were made based on data from lunar eclipses.

Note. Eratosthenes is the first to attempt a scientific chronology starting with the siege of Troy (around 1190 BCE). His *Geographica* consisted of three books, with a history of geography and a consideration of mathematical geography. He must have had a great reputation, because we know that Archimedes dedicated his *The Method* to Eratosthenes. His peers gave him the nickname "Beta," because they interpreted him as the second-best in so many different areas (though Eves on page 170 mentions that another interpretation of the nickname is that he was never the *best* in any of his many areas).

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