6.4. Apollonius

Note. In this section, we concentrate on Apollonius of Perga's main work, *Treatise on Conic Sections*. We also consider other works of Apollonius (though in much less detail). The sources for this section, in addition to Eves, are the MacTutor biography webpage for Apollonius, Thomas Heath's *A History of Greek Mathematics*, Volume 2 (Oxford: Clarendon Press, 1921), pages 126 to 196, and Heath's *Apollonius of Perga, Treatise on Conic Sections* (Cambridge University Press, 1896).

Note 6.4.A. We met Menaechmus (circa 380 BCE–circa 320 BCE) in Section 4.5. Duplication of the Cube where we saw that, using conic sections, he gave two solutions to the duplication of the cube problem (see Note 4.5.B). See also Note 3.A in my online notes for the historical component of Introduction to Modern Geometry (MATH 4157/5157) on Chapter 3. Conic Sections. As stated in Heath's *Apollonius of Perga, Treatise on Conic Sections*: "Thus the evidence so far shows (1) Menaechmus (a pupil of Eudoxus and a contemporary of Plato) was the discoverer of the conic sections, and (2) that he used them as a means of solving the problem of the doubling of the cube." (See his page xix.) According to the MacTutor biography webpage of Menaechmus (accessed 9/16/2023): "Menaechmus is famed for his discovery of the conic sections and he was the first to show that ellipses, parabolas, and hyperbolas are obtained by cutting a cone in a plane not parallel to the base." However, the key figure in the history of the conic sections is Apollonius of Perga (circa 262 BCE–circa 190 BCE). It is Apollonius in his *Treatise on Conic Sections* (or simply *Conics*) that the three types of conic sections get their names.

Note 6.4.B. Eves describes Euclid, Archimedes, and Apollonius as "the three great mathematical giants of the third century B.C." (See his page 171.) Very little is known about his life. It is known that he was born in Perga in southern Asia Minor (in modern day Turkey). When he was young he went to Alexandria and studied with the successors of Euclid, and later visited Pergamum where met "Eudemus of Pergamum" to whom he dedicated the first two books of the *Conics*. The strength of the *Conics* earned Apollonius the name "The Great Geometer" among his contemporaries. *Treatise on Conic Sections* consists of eight books and about 400 propositions. It thoroughly explores the three conic sections, superseding previous work on the topic by Menaechmus (circa 380 BCE–circa 320 BCE; mentioned above in Note 6.4.A), Aristaeus (circa 370 BCE–circa 300 BCE; in his now lost *Five Books concerning Solid Loci*), and Euclid (circa 325 BCE–circa 265 BCE; in his lost work *Conics*, see Note 5.8.F in Section 5.8. Euclid's Other Works).

Image from the Wikipedia webpage on Apollonius (accessed 9/16/2023). An illustration from the 1537 edition of Apollonius' works.

The primary characters in the early history of conic sections are (1) Menaechmus, who defined conic sections as intersections of cones with planes (literally "conic sections"), (2) Apollonius who defined the conic sections in terms of the areas of a square and a rectangle determined by points on the conic section, and (3) Pappus who defined the conic sections in terms of a directrix, focus, and eccentricity (we'll see more of Pappus in Section 6.9. Pappus).

Note 6.4.C. Pappus (circa 290 CE–circa 350 CE) wrote lemmas to the *Conics*, the geometer and commentator Serenus (circa 300 CE–circa 360 CE) wrote a commentary, and sources indicate that Hypatia (circa 370 CE–March 415 CE) also wrote a commentary on it. Eutocius of Ascalon (circa 480 CE–circa 540 CE) prepared an edition of the first four Books and wrote on commentary on these. Today, on the first four Books survive in Greek. Books V–VII still exist in Arabic, but Book VIII is lost. Heath in *A History of Greek Mathematics* (Volume 2, pages 127 and 128) tells us:

> "A Latin translation of Books I–IV was published by Joyannes Baptista Memus at Venice in 1537 [presumably the source of the image of Apollonius given above]; but the first important edition was the translation by Commandinus (Bologna, 1566), which included the lemmas of Pappus and the commentary of Eurocius, and was the first attempt to make the book intelligible by means of explanatory notes. . . . The first published version of Books V–VII was a Latin translation by Abraham Echellensis and Giacomo Alfonso Borelli (Florence, 1661). . . The Greek text of Books I–IV is now available, with the commentaries of Eutocius, the fragments of Apollonius, &c., in the definitive edition of Heiberg (Teubner, 1891–3)."

We mentioned Johan Heiberg in Section 5.3. Euclid's "Elements"; see Note 5.3.J. Heath, in fact, wrote his own version of the *Conics* which appeared in 1896 as: *Apollonus of Perga: Treatise on Conic Sections*, Edited in Modern Notation with Introductions Including as Essay on the Earlier History of the Subject (Cambridge University Press, 1896). The Preface and Introduction of this book run 170 pages. This book is still in print.

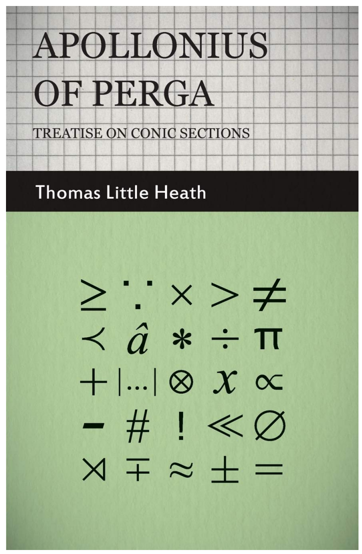


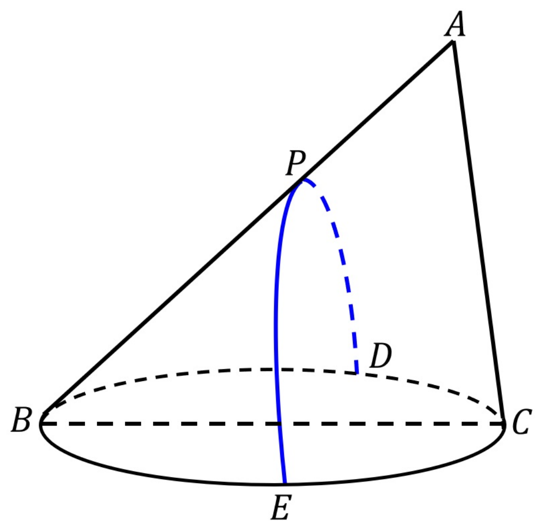
Image of the Carruthers Press (2015) printing

from Amazon.com (accessed 9/16/2023)

Heath mentions four main works that he used in his preparation of his book. Two are by Edmund Halley, one is by Heiberg (as mentioned in the quote above), and the other is a reproduction in German of the *Conics* by H. Balsam (Berlin, 1861). Heath praises this last work for its explanatory notes and collection of around 400 figures given at the end of the book.

Note 6.4.D. We now describe the Books of the *Treatise on Conic Sections* using the prefaces for the books as written by Apollonius himself (see Heath's *A History* *of Greek Mathematics* Volume 2, pages 128 to 132), as well as Eve's descriptions. The first four books give an elementary introduction. Book I gives the means of producing the three conic sections (including the two branches of the hyperbola; Menaechmus only considered one branch of the hyperbola). Book II contains the properties of the diameters and the axes, as well as asymptotes and tangents. The third book includes theorems concerning intersections of chords and tangents to a conic section. The (optical) focal properties of the "central" conics (i.e., the ellipse and the hyperbola) are given near the end of Book III. Book IV proves the converses of some of the propositions of Book III and considers intersections of conic sections. "Book V is the most remarkable and original of the extant books" (Eves, page 173). Book V considers maximum and minimum line segments drawn from a given point to the conic section and considers the construction and enumeration of such line segments. Book VI contains constructions concerning similar conics, and how to find a given conic as a section of a given cone. Book VII contains propositions concerning diameters of conic sections and figures (such as rectangles and parallelograms) determined by them.

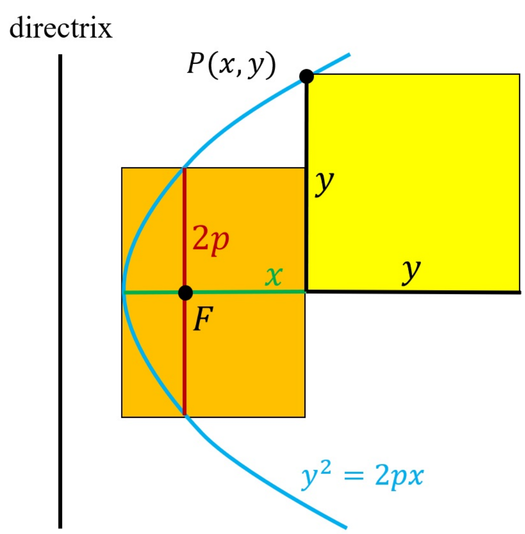
Note 6.4.E. Apollonius considers double circular cones in his generation of conic sections. However, he does not restrict himself to right circular cones, like is often done in the introduction of conic sections today. For example, in Calculus 3 (MATH 2110) right circular cones are used; see my online notes for Calculus 3 on Section 11.6. Conic Sections (notice Figure 11.36). Apollonius considers a given circle and any point outside the plane of the circle and not necessarily lying on the straight line through the center of the circle perpendicular to its plane. A straight line passing through the point is made to move, while always passing through the fixed point, so as to pass through all the points of the circle (this wording is Heath's from his *History*, Volume 2 page 134). This gives the double cones needed. When the cones are not right cones (so that the fixed point is not on the described perpendicular line), they are *oblique* or, as Apollonius puts is, "scalene." The conic sections can then be taken by by cutting the double cone with a plane (1) parallel to one of the lines through the fixed point and a point on the given circle (this gives a parabola; see the figure below), (2) that intersects both of the cones (this gives a hyperbola), or (3) intersects just one cone (this gives an ellipse). With the plane of the given circle in a horizontal plane, these three cases can be related to the "steepness" of the cutting plane.



In this figure, the "given circle" is the one containing points B C D E, and the "fixed point" is point A. The curve given in blue is a parabola because the plane containing it, that is the plane determined by points D E P, is parallel to the line through A and C.

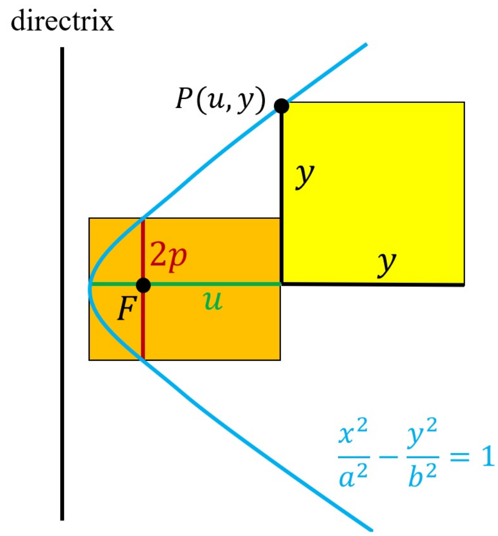
Note 6.4.F. In order to understand Apollonius' choices for the names of the conic sections, we introduce a coordinate system and describe areas of rectangles in terms of these coordinates and the location of the focus. Of course this is not Apollonius' approach since coordinate geometry does not appear until Descartes introduces it. However, Apollonius does refer to areas of rectangles. In Heath's translation of the *Conics* (page 9), we have the following (where we have changed his labels of points so that they agree with the picture below):

> It follows that the square on any ordinate y to the axis of the parabola [Apollonius calls the axis the "fixed diameter"] is equal to a rectangle applied ($\pi\alpha\rho\alpha\beta\acute{\alpha}\lambda\lambda\epsilon\iota\nu$) to the fixed straight line of length $2p$ [the latus rectum] to the fixed straight line drawn at right angles to the axis of the parabola, with altitude equal to the corresponding abscissa x. Hence the section is called a PARABOLA.



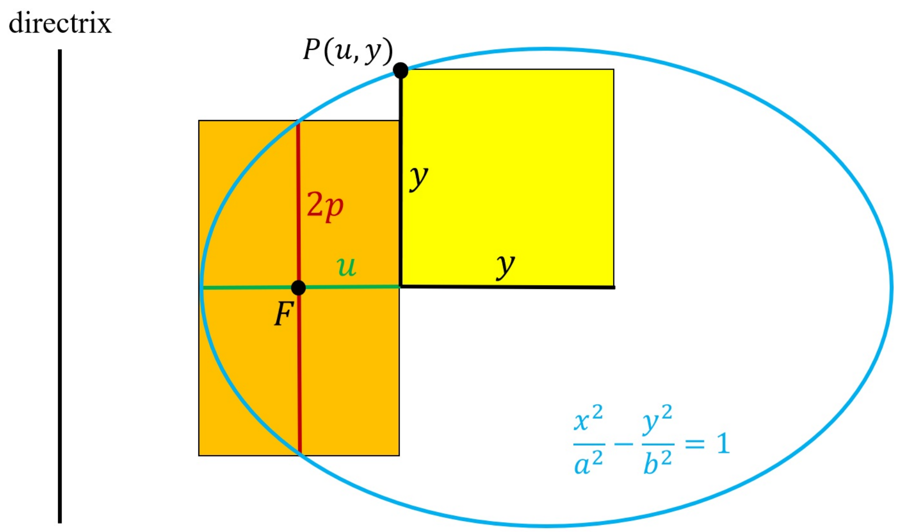
Here the prefix *para* is being used because the area of the square is the same as the area of the rectangle (in Greek *para* means "next to" or "side by side" reflecting a sameness). For now, ignore the directrix in the figure. On page 10 of Heath's translation we have:

> It follows that the square on the ordinate (that is, the square with area y^2) is greater than a rectangle whose height is equal to the latus rectum $2p$ and whose base is the abscissa x. [Apollonius actually speaks of the equality of the square and a rectangle whose base overlaps the square by a certain amount. In this way he is dealing with an equation of the form $y^2 = px + (p/d)x^2$.] Hence the section is called a HYPERBOLA.



Here the prefix *hyper* is being used because the area of the square is greater than the area of the rectangle (in Greek *hyper* means "over" or "beyond," reflecting the inequality). On page 12 of Heath's translation we have:

> Thus the square on the ordinate (that is, the square with area y^2) is less than a rectangle whose height is equal to the latus rectum $2p$ and whose base is the abscissa x. [Apollonius actually speaks of the equality of the square and a rectangle whose height falls short of the latus rectum by a certain amount. In this way he is dealing with an equation of the form $y^2 = px - (p/d)x^2$.] The section is therefore called an ELLIPSE.



Here the name *ellipse* is being used because the area of the square is less than the area of the rectangle (in Greek, *elleipsis* means "falling short" or "defect," reflecting the inequality). Apollonius introduces a method for finding the *latus rectum* (which is the red line segment of length $2p$ in each of the figures above). In this way, he can relate the areas of the square, y^2, and the area of the rectangle, either $2px$ or $2pu$, without the need for coordinates. The material of this note is also presented in Introduction to Modern Geometry (MATH 4157/5157) in Section 3.1. The Parabola, Section 3.2. The Ellipse, and Section 3.3. The Hyperbola.

Note 6.4.G. The focal properties of the hyperbola and ellipse are shown in Book III of the *Conics*. Surprisingly, "[t]he focus of a *parabola* is not used or mentioned by Apollonius" (Heath's translation of *Treatise on Conic Sections*, page 114). However, Apollonius likely wrote on the reflective property of the parabola in his lost work *On the Burning-Mirror* (see Note 5.4.J below). Apollonius does address the foci of the hyperbola and ellipse (the so-called "central conics"). This is done in

Book III, Proposition 48 (which appears as Proposition 71 on page 116 of Heath's translation). It is stated as:

Book III, Proposition 48. The focal distances of P makes equal angles with the tangent at that point.

The "focal distances" are the line segments joining point P on the central conic to each of the two foci. This is illustrated for the hyperbola in the following figure (left) and for the ellipse (right). The figure for the hyperbola is from Alexander Ostermann and Gerhard Wanner's *Geometry by Its History*, Undergraduate Texts in Mathematics (Springer Verlag, 2012); this is the text used for the historical component of Introduction to Modern Geometry (MATH 4157/5157).

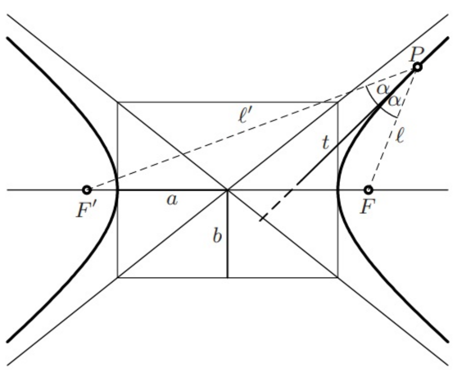
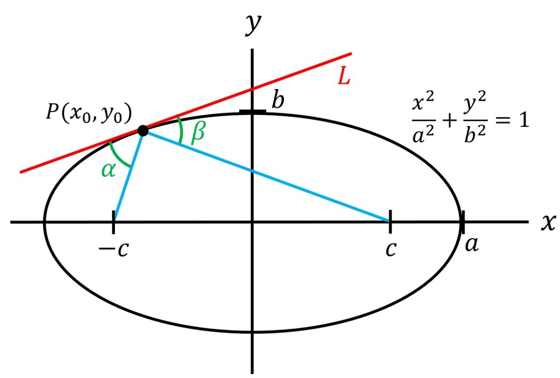
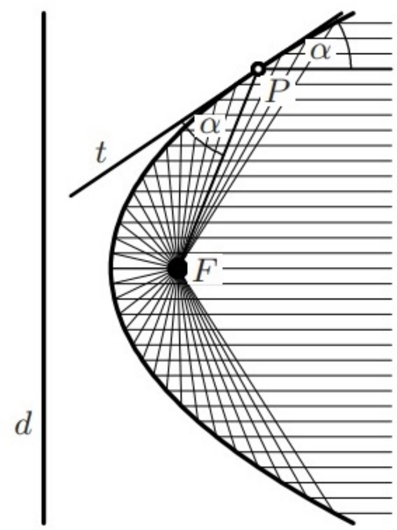


Fig. 3.11. A hyperbola and its tangent



Notice that this property is related to the optical interpretation of "the angle of incidence equals the angle of reflection" (in the figure for the ellipse, $\alpha = \beta$). In fact, this result also holds for the parabola. However, we must interpret the location of the second focus "at infinity." In this way, the we take one of the focal distances as a half-line inside the parabola which is parallel to the axis of the parabola and ends at point P. This is illustrated below in part of Figure 3.1 from *Geometry by Its History*. Each of the reflective properties are easily proved using calculus (often as exercises), where tangent lines are found using differentiation. In fact, such solutions to these exercises are given in my online notes for Introduction to Modern Geometry (MATH 4157/5157), mentioned above, for the parabola (see Exercise 11.6.81, from Calculus 3 MATH 2110, in Section 3.1. The Parabola), the ellipse (see Note 3.2.C in Section 3.2. The Ellipse), and the hyperbola (see Note 3.3.C in Section 3.3. The Hyperbola).



Note 6.4.H. You are likely familiar with the definition of an ellipse as the set of points in a plane whose distances from two fixed points in the plane have a constant sum, and the definition of a hyperbola as the set of points in a plane whose distances from two fixed points in the plane have a constant difference. This is the definition taken in Calculus 3 (MATH 2110); see my online Calculus 3 notes on Section 11.6. Conic Sections. In Calculus 3 a parabola is defined in terms of distances from a focus an a directrix (as we'll see in Section 6.9. Pappus, that Pappus [circa 290 CE–circa 350 CE] used a focus and a directrix to define each of the conic sections). Either of these approaches allow us to derive the formulas of the conic sections in terms of Cartesian coordinates. Apollonius agrees with the Calculus 3 definitions

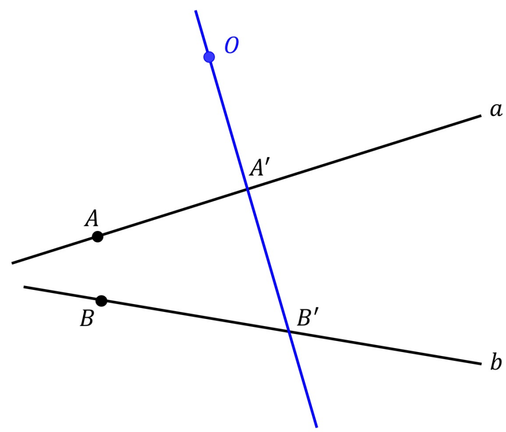
for the "central conics," as he shows in Book III, Propositions 51 and 52 which together state (quoting from Heath's translation of the *Conics*):

Book III, Propositions 51 and 52. In an ellipse the sum, and in a hyperbola the difference, of the focal distances of any point is equal to the axis AA'.

The "axis AA'" is the major axis of the ellipse, and the line segment joining the two vertices of a hyperbola (these line segments are of length $2a$ in the figures of an ellipse and a hyperbola given above in Note 6.4.G). As suggested by the figure given in Note 6.4.E for a parabola as a conic section, which conic section produced by the intersection of a plane with a cone is dependent on how "steep" the plane is relative to the sides of the cone. This steepness can be addressed in terms of "Dandelin spheres." The original work on this is due to Germinal Dandelin (April 12, 1794–February 15, 1847) and presented in his "Memoir on some remarkable properties of the parabolic focale [i.e., oblique strophoid]," *Nouveaux mémoires de l'Académie royale des sciences et belles-lettres de Bruxelles* (in French), **2**, 171-200 (1822). Credit for this approach is also sometimes given to Adolphe Quetelet based on his "Inaugural mathematical dissertation on some geometric loci and also focal curves," doctoral thesis (University of Ghent, Belgium, 1819). This is explained (in the case of right cones) in the history component of Introduction to Modern Geometry (MATH 4157/5157), where some of Apollonius' propositions are restated in terms of steepness of the plane as compared to the slope of the sides of the cone. See Theorem 3.1 of Section 3.1. The Parabola (where Apollonius' Proposition I.11 is restated this way), Theorem 3.2.A of Section 3.2. The Ellipse (where Apollonius' Proposition III.52 is restated and combined with the "constant sum of distances from two foci" idea is also given), and Theorem 3.3.A of Section 3.3. The Hyperbola (where Apollonis' Proposition III.51 is restated and combined with the "constant difference of distances from two foci" idea is also given).

Note. Eves states (on his page 174): "Even from the above brief sketch of contents, we see that the treatise is considerably more complete then the usual present-day college course in the subject." However, Heath observes (see page vii of his translation of the *Conics*) that "the influence of Apollonius upon modern text-books on conic sections is, so far as form and method are concerned, practically nil." This is not surprising since today conic sections are taught using Cartesian coordinates and algebraic equations, and these were not developed until the 1600s (as will be discussed in Section 10.1. Analytic Geometry and Section 10.2. Descartes).

Note 6.5.I. Pappus (circa 290 CE–circa 350 CE), in his *Mathematical Collection* Book VII, mentions six other works of Apollonius which formed part of his *Treasury of Analysis.* Of the six, the only one that survives is *On the Cutting-off of a Ratio* (Eves calls this *On Proportional Sections*). It survives in Arabic and was translated into Latin by Edmund Halley in 1706. The general problem it addresses is (as Eves states it): "Given two lines a and b [parallel to one another of intersecting] with the fixed points A on a and B on b, to draw through a given point O a line $OA'B'$, cutting a in A' and b in B' so that $AA'/BB' = k$, a given constant." See the figure below, based on Eves' Figure 47. The work *On the Cutting-off of an Area* (Eves calls this *On Spatial Sections*) dealt with a similar to problem to that just mentioned, except that the intercepts on the given straight lines are require not to have a given ratio, but instead to have a given product (where the product is dealt with in terms of the area of a rectangle).



Edmund Halley (November 8, 1656–January 14, 1742) attempted a restoration of this work in his edition of the *De sectione rationis*, which was published in German in 1824 (Berlin: Georg Reimer); this can be viewed online on the HathiTrust.org website (accessed 9/20/2023). The work *On Determinate Section*, which was exhaustively discussed as evidenced by Pappus' account of the contents of this work in his *Collection* Book VII, concerned the general problem: "Given four points A, B, C, D on a straight line, to determine another point P on the same straight line such that the ratio $(AP)(CP) : (BP)(DP)$ has a given value." The work *On Contact* or *Tangencies* deals with the problem (again, as is known from Pappus' *Collection*): "Given three things, each of which may be either a point, a straight line or a circle, to draw a circle which shall pass through each of the given points (so far as it is points that are given) and touch [i.e., be tangent to] the straight lines or circles." This problem is now known as the "Problem of Apollonius." The work *Plane Loci* is also known through a thorough account in Pappus' *Collection*. So complete is Pappus' description, that restorations of this work have been given by Pierre de Fermat (August 17, 1601–January 12, 1665), Frans van Schooten (May 15, 1615–May 29, 1660), and (most completely) Robert Simson (October 14, 1687–October

1, 1768); we first saw mention of Robert Simpson in connection with Latin and English editions of Euclid's *Elements* (see Note 5.3.H in 5.3. Euclid's "Elements"). Eves calls attention to two theorems in *Plane Loci* (see Eves' page 175):

1. If A and B are fixed points and k a given constant, then the locus of a point P, such that $AP/BP = k$, is either a circle (if $k \neq 1$) or a straight line (if $k = 1$).

2. If $A, B, \ldots$ are fixed points and $a, b, \ldots, k$ are given constants, then the locus of a point P, such that $a(AP)^2 + b(BP)^2 + \cdots = k$, is a circle.

The circle in the first theorem is called the "Circle of Apollonius." The Apollonius circle is also seen in Axiomatic and Transformational Geometry (MATH 5330) in Section 52. Möbius Transformations. The work *Vergings* or *Inclinations* addresses the problem of placing between two straight lines, a straight line and a curve, or two curves, a straight line [segment] of given length in such a way that it *verges* towards a fixed point (that is, if extended then it will pass through the fixed point). The source for this note (in addition to Eves) is Thomas Heath's *A History of Greek Mathematics, Volume 2* (Oxford: Clarendon Press, 1921), pages 175–192.

Note 6.4.J. Other lost works of Apollonius are also known. We know of *A Comparison of the Dodecahedron with the Icosahedron* is mentioned by Hypsicles (circa 190 BCE–circa 120 BCE) in the preface to his "Book XIV" of Euclid's *Elements* (see Note 5.3.A in Section 5.3. Euclid's "Elements"). It includes a proof that for a dodecahedron and icosahedron inscribed in the same circle, the ratio of their surface areas is the same as the ratio of their volumes. In *General Treatise* Apollonius dealt with fundamental principle of mathematics, such as definitions (in particular, the definitions of line, plane, and solid angle are elucidated), axioms, etc. The work *On the Cochlias* considers the cylindrical helix. Apollonius offers an extension of Euclid's theory of irrationals in *Unordered Irrationals*. Based on reference to *On the Burning-Mirror* by other, it is suspected that Apollonius knew of the reflective properties of parabolic mirrors. In Heath's *A History of Greek Mathematics, Volume 2*, he states on page 194:

> "...we can well believe that the parabolic form of mirror was also considered in Apollonius's work, and that he was fully aware of the focal properties of the parabola, notwithstanding the omission from the *Conics* of all mention of the focus of a parabola."

We mentioned this claim above in Note 6.4.G. In Eutocius' (circa 480 CE–circa 540 CE) commentary on Archimedes' *Measurement of the Circle*, the work *Quick Delivery* by Apollonius is referenced as including an approximation of the value of π in a different calculation from that of Archimedes and an improvement on Archimedes' approximation. The source for this note (in addition to Eves) is Thomas Heath's *A History of Greek Mathematics, Volume 2* (Oxford: Clarendon Press, 1921), pages 192–194.

Revised: 9/21/2023